Extracting resonance information from lattice QCD

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Resolving scattering in lattice QCD is not straightforward

Lattice QCD can determine (finite volume) energy eigenstates

**Problem:**

Energy eigenstates $\rightarrow$ scattering parameters?

Lüscher quantisation condition

Finite-volume Hamiltonian

Multi-channel systems $(\pi\pi \leftrightarrow K\bar{K})$

Lambda-1405
QCD has a rich spectrum
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Spectrum in lattice QCD
In the good old days

Lattice QCD was forced to use large quark masses

Low-lying spectrum comprised of stable particles

[Curves are just a sketch: not a fit]
In the good old days

Lattice QCD was forced to use large quark masses
Low-lying spectrum comprised of stable particles
[Curves are just a sketch: *not a fit*]
Baryon mass: “Sketching a curve”

\[ M_B = M_B^0 \left( \frac{1}{1 + \frac{3}{2} \frac{m_\pi}{M_B^0}} + \frac{3}{2} \frac{m_\pi}{M_B^0} \right) \]
Baryon mass: “Sketching a curve”

\[ M_B = M_B^0 \left( \frac{1}{1 + \frac{3}{2} \frac{m_\pi}{M_B^0}} + \frac{3}{2} \frac{m_\pi}{M_B^0} \right) \]

One parameter = magnitude, slope & curvature!
State-of-the-art LQCD spectrum

$1/2^+:$ “Free-particle” energies on a 2.9 fm box

[Interactions will shift energy levels]
State-of-the-art LQCD spectrum

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“Data”: Mahbub et al. [CSSM] PLB(2012)
State-of-the-art LQCD spectrum

1/2⁺: “Free-particle” energies on a 2.9 fm box
[Interactions will shift energy levels]

“N + π”
“N + ππ”

“Data”: Mahbub et al. [CSSM] PLB(2012)
“N + π”
“N + ππ”
“∆ + π”

State-of-the-art LQCD spectrum

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State-of-the-art LQCD spectrum

$1/2^+$: “Free-particle” energies on a 2.9 fm box
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“Data”: Mahbub et al. [CSSM] PLB(2012)
Resonance spectrum is no longer separated from multiparticle thresholds
Resonance parameters?

Lüscher: 2-body elastic “scattering”, one can map finite-volume energies directly to phase shifts

Map out volume dependence of energy levels
Scattering parameters?

Finite volume spectra in lattice QCD encode a quantum mechanical admixture of scattering states and resonance-like structures.

How can we use finite-volume spectra to identify scattering parameters (eg. S-matrix)?
Lüscher: Qualitative (1-D QM)

1-D Scattering of 2 identical bosons; finite-range potential

General solution (non-interacting region)

Right $\psi^{(r)}(x) = Ae^{-ikx} + Be^{ikx}$

[Left by Bose symmetry]

Steady-state: conservation of probability $\Rightarrow |B| = |A|$

Define $\frac{B}{A} = e^{i2\delta} \Rightarrow \psi^{(r)}(x) = A \left( e^{-ikx} + e^{i2\delta(k)} e^{ikx} \right)$

For any potential, scattering process is entirely determined by the phase shift $\delta(k)$
Lüscher: Qualitative (1-D QM)

**Periodic boundary conditions**

"Box" length $L$, $\psi(x + L) = \psi(x)$

**Boundary** $x = \pm L/2$

Continuous: Bose symmetry  \[ \Rightarrow \psi^{(r)}(L/2) = \psi^{(l)}(-L/2) \]

Smooth \[ -ik \left( Ae^{ikL/2} - Be^{-ikL/2} \right) = ik \left( Ae^{ikL/2} - Be^{-ikL/2} \right), \]
\[ \Rightarrow \frac{B}{A} = e^{-ikL}. \]

**Potential defines phase shift**

**Eigenvalue equation**

\[ e^{i2\delta(k)} - e^{-ikL} = 0 \]

Scattering phase  Lattice geometry
For 3-D, we require general solution for free particles outside spherical potential region with cubic periodicity.
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Lüscher (3-D)

Eigenvalue equation

$$\det \left\{ U_{l';m';l,m} - e^{i2\delta_{l'} \delta_{l,l'} \delta_{m,m'}} \right\} = 0$$

Lattice geometry  Scattering phases

Finite volume spectrum uniquely determined in terms of on-shell scattering parameters

up to quantum fluctuations  $\sim \exp(-m_\pi L)$
Example

\[ \Delta \leftrightarrow N \pi \]
Dynamical model \( \Delta \rightarrow N\pi \)

Low-energy interaction based on chiral EFT

Dynamically dressed

\[
\begin{align*}
\pi & \quad \Delta \\
N & \quad \Delta
\end{align*}
\]

\[
\begin{align*}
\pi & \\
N & \Delta
\end{align*}
\]

\[
\begin{align*}
\Delta & \quad N \\
\Delta & \quad \Delta
\end{align*}
\]

\[
\begin{align*}
\text{Re} \, \delta \, (\text{degrees})
\end{align*}
\]

\[
\begin{align*}
E \, (\text{GeV})
\end{align*}
\]
Finite-volume spectrum

Solid lines are interacting finite-volume energy levels
Finite-volume spectrum

Solid lines are interacting finite-volume energy levels

Measure any discrete lattice eigenvalue \( \Rightarrow \) identify phase shift at that energy
Finite-volume Hamiltonian

Same boundary condition as Lüscher formulation

Construct a basis of non-interacting Fourier modes satisfying the peridocity of the lattice

\[ \{ \Delta, (N \pi)_1, (N \pi)_2, (N \pi)_3, \ldots \} \]

Bare Hamiltonian

\[ H_0|\Delta\rangle = \Delta^0|\Delta\rangle; \quad H_0|(N \pi)_j\rangle = \sqrt{m_\pi^2 + k_j^2}|(N \pi)_j\rangle \]

\[ k_j^2 = (n_1^2 + n_2^2 + n_3^2) \left( \frac{2\pi}{L} \right)^2 \]

Interaction

\[ \langle \Delta|H_I|(N \pi)_j\rangle \sim g(k_j) \]

\[ \pi \]

\[ N \]
Finite-volume Hamiltonian

"Resonance-like" state

Construct a basis of non-interacting Fourier modes satisfying the periodicity of the lattice

$$\{\Delta, (N\pi)_1, (N\pi)_2, (N\pi)_3, \ldots\}$$

Bare Hamiltonian

$$H_0|\Delta\rangle = \Delta^0|\Delta\rangle; \quad H_0|(N\pi)_j\rangle = \sqrt{m_\pi^2 + k_j^2}|(N\pi)_j\rangle$$

Interaction

$$\langle \Delta | H_I | (N\pi)_j \rangle \sim g(k_j) \frac{\pi}{L}$$
Finite-volume Hamiltonian

Construct a basis of non-interacting Fourier modes satisfying the periodicity of the lattice

\[ \{ \Delta, (N\pi)_1, (N\pi)_2, (N\pi)_3, \ldots \} \]

Bare Hamiltonian

\[ H_0 |\Delta\rangle = \Delta^0 |\Delta\rangle; \quad H_0 |(N\pi)_j\rangle = \sqrt{m^2_\pi + k^2_j} |(N\pi)_j\rangle \]

Interaction

\[ \langle \Delta | H_I |(N\pi)_j\rangle \sim g(k_j) \]

“Resonance-like” state

“Scattering” states

on as Lüscher
Discrete Hamiltonian

“Bare” Hamiltonian

\[
H_0 = \begin{pmatrix}
\Delta_0 & 0 & 0 & \cdots \\
0 & \omega_\pi(k_1) & 0 & \cdots \\
0 & 0 & \omega_\pi(k_2) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

Interaction Hamiltonian

\[
H_I = \begin{pmatrix}
0 & g_{\Delta N}^{\text{fin}}(k_1) & g_{\Delta N}^{\text{fin}}(k_2) & \cdots \\
g_{\Delta N}^{\text{fin}}(k_1) & 0 & 0 & \cdots \\
g_{\Delta N}^{\text{fin}}(k_2) & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

Lattice spectrum determined by eigenvalue equation

\[
\text{det}(H - \lambda I) = 0
\]
Hamiltonian spectrum
Identical to Lüscher (to the eye)
Resonance parameter estimation

On each volume determine phase shift from Lüscher equation
Use Breit-Wigner to interpolate points and extract mass and width
OR  Directly fit Hamiltonian parameters to spectrum
Resonance mass

Lüscher & Hamiltonian approach both give a reliable estimate of resonance mass at finite $L$. 
Inelastic channels
Extension to coupled channel

Extension of Lüscher by He, Feng & Liu JHEP(2005)

\[
S^{(l)}(E) = \begin{pmatrix}
\eta_l e^{2i\delta_1^l} & i\sqrt{1 - \eta_l^2} e^{i(\delta_1^l + \delta_2^l)} \\
i\sqrt{1 - \eta_l^2} e^{i(\delta_1^l + \delta_2^l)} & \eta_l e^{2i\delta_2^l}
\end{pmatrix}
\]

Scattering parameters as a function of E

\(\delta_1, \delta_2, \eta\)

Finite-volume eigenvalue equation (s-wave, no L-mixing)

\[
\cos(\Delta_1 + \Delta_2 - \delta_1^0 - \delta_2^0) = \eta \cos(\Delta_1 - \Delta_2 - \delta_1^0 + \delta_2^0)
\]

\[
\Delta_i = M_{00;00}(q_i^2) = \frac{Z_{00}(1; q_i^2)}{\pi^{3/2} q_i}
\]
$J=0$ Isoscalar $\pi\pi$, $K\bar{K}$

Phase shift

Inelasticity

Wu, Lee, Thomas, RDY, arXiv:1402.4868
Hamiltonian spectrum

\[ \{ \sigma, (\pi\pi)_0, (\pi\pi)_1, (\pi\pi)_2, \ldots, (K\bar{K})_0, (K\bar{K})_1, (K\bar{K})_2, \ldots \} \]

Dashed lines are non-interacting levels
FV Spectrum “Prediction”

Lüsher: red dots
Hamiltonian: black curves
Same quantisation condition ⇒ same spectrum!
Applications for lattice QCD, we want the inverse:

*ie.* $Spectrum \rightarrow Phase \ shifts$
Inversion
Single channel: straightforward

Inversion
Single channel: straightforward

2 channels: requires degeneracy at independent L

Inversion
Not quite that bad: Boosted frames
Modified boundary condition gives independent constraint
\[ \vec{P} = 0 : \]

\[ E_2 = 2m, 2\sqrt{m^2 + q^2}, 2\sqrt{m^2 + 2q^2}, \ldots \]

\[ q = \frac{2\pi}{L} \]

\[ \vec{P} = (1, 0, 0)q : \]

\[ E_2 = m + \sqrt{m^2 + q^2}, \sqrt{m^2 + q^2} + \sqrt{m^2 + 2q^2}, \ldots \]


**Not quite that bad: Boosted frames**

Modified boundary condition gives independent constraint
"Pointwise" determination from BCs alone
⇒ parameterisation essential for “complete” extraction
Boosted momentum frames

Lüscher: red dots
Hamiltonian: black curves
Same quantisation condition $\Rightarrow$ same spectrum!
Inversion with Hamiltonian

Requires a choice of Hamiltonian

<table>
<thead>
<tr>
<th></th>
<th>Model A</th>
<th>Model B</th>
<th>Model C</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma )</td>
<td>( \frac{g_{\sigma,\alpha}}{1 + (c_{\alpha} k)^2} )</td>
<td>( \frac{g_{\sigma,\alpha}}{(1 + (c_{\alpha} k)^2)^2} )</td>
<td>( g_{\sigma,\alpha} e^{-(c_{\alpha} k)^2} )</td>
</tr>
<tr>
<td>( \pi, K )</td>
<td>( \pi, K )</td>
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<td>( \pi, K )</td>
</tr>
<tr>
<td>( \pi, \overline{K} )</td>
<td>( \frac{G_{\sigma,\alpha}}{(1 + (d_{\alpha} k)^2)(1 + (d_{\beta} k)^2)} )</td>
<td>( \frac{G_{\sigma,\alpha}}{(1 + (d_{\alpha} k)^2)^2(1 + (d_{\beta} k)^2)^2} )</td>
<td>( G_{\alpha,\beta} e^{-(d_{\alpha} k)^2} e^{-(d_{\beta} k)^2} )</td>
</tr>
</tbody>
</table>

\( \alpha, \beta = (\pi \pi), (K \overline{K}) \)
Inversion with Hamiltonian
Just 2 box sizes (8 eigenstates)
S-matrix rather well reproduced: Independent of Hamiltonian form

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Just 2 box sizes (8 eigenstates)
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S-matrix rather well reproduced: Independent of Hamiltonian form

Break down in region without lattice eigenstates
Inversion with Hamiltonian
Just 2 box sizes (12 eigenstates)
Hamiltonian formulation offers a technique to parameterise interaction and directly fit to lattice eigenstates

Inversion with Hamiltonian
Just 2 box sizes (12 eigenstates)
Lambda-1405 Lattice simulation

Stable at heavy quark masses

Numerical “data”

\( \pi \Sigma \) threshold
Effective Hamiltonian

Flavour singlet Lambda couples to meson baryon states

Away from SU(3)-flavour symmetric point, 4-channel system

\[ \phi B = \{ \pi \Sigma, \bar{K}N, \eta \Lambda_8, K \Xi \} \]

Coupling fixed from physical decay width \( \Lambda \to \pi \Sigma \)

Bare mass dependent only on singlet quark mass (to leading order)

\[ m_{\Lambda}^{\text{bare}} = m_0 + \alpha (m_K^2 + m_\pi^2 / 2) \]
Fit to lattice eigenstates

Solid curves show mass dependence of finite-volume eigenstates
Dashed lines denote non-interacting 2-body states
Real part of pole position
Inflections at decay thresholds
Real part of pole position

Inflections at decay thresholds
Lattice apologies

Technical challenges of getting spectrum of eigenstates

Partial wave mixing on the finite volume

  Full rotation group broken by the boundary conditions

Exponentially suppressed energy corrections

  Interactions of field fluctuations with boundary  $\sim \exp(-m_\pi L)$

>2-particle inelastic thresholds

  3-particle channels & above still requires more work
Summary

For elastic 2-body scattering, the boundary quantisation uniquely predicts the finite-volume spectra in terms of the on-shell scattering amplitude.

For coupled-channel systems, Hamiltonian approach appears to offer a useful parameterisation of S-matrix.

Hamiltonian formulation offers a natural method to extend to more complicated systems, e.g., many-body systems; transition matrix elements.

Evidence for a Lambda-1405 in numerical lattice simulation.
Using boosted frames, can extract the scattering phases at many more eigenenergies (momenta)
Eigenvectors

Strength of coupling of bare Delta to energy eigenstates.
Clearly no distinction between “local” and scattering state (within width of resonance)
Bootstrap outcomes

Lambda pole position distribution
KKbar phase shift not determined experimentally

Application?
KKbar phase shift not determined experimentally

Lattice eigenstates of ~50 MeV precision could discriminate model dependence