An Introduction to Quantum Field Theory

Hartmut Wittig

Theory Group
Deutsches Elektronen-Synchrotron, DESY
Notkestrasse 85
22603 Hamburg
Germany

Lectures presented at the School for Young High Energy Physicists,
Rutherford Appleton Laboratory, September 2003
Contents

0 Prologue

1 Ingredients of Quantum Field Theory
1.1 Basic concepts of Quantum Mechanics .......................... 3
1.2 Special Relativity ........................................ 5
1.3 Relativistic wave equation: The Klein-Gordon equation .......... 6
Problems .................................................. 7

2 Quantisation of the free scalar field .......................... 8
2.1 Heisenberg picture ....................................... 8
2.2 Plane wave solutions of the Klein-Gordon equation ............ 9
2.3 Quantisation of the real Klein-Gordon field .................... 10
2.4 Causality and commutation relations ......................... 12
Problems .................................................. 16

3 Lagrangian formalism ........................................ 17
3.1 Classical Mechanics ...................................... 17
3.2 Classical field theory .................................... 19
3.3 Quantum field theory .................................... 22
3.4 Summary: canonical quantisation for real scalar fields ....... 24
Problems .................................................. 25

4 Interacting scalar fields ...................................... 26
4.1 The $S$-matrix ........................................ 27
4.2 More on time evolution: Dirac picture ....................... 28
4.3 $S$-matrix and Green’s functions .......................... 31
Problems .................................................. 34

5 Perturbation Theory ........................................ 34
5.1 Wick’s Theorem ........................................ 36
5.2 The Feynman propagator .................................. 37
5.3 Two-particle scattering to $O(\lambda)$ .......................... 38
5.4 Graphical representation of the Wick expansion: Feynman rules .. 40
5.5 Feynman rules in momentum space ........................ 42
5.6 $S$-matrix and truncated Green’s functions .................. 43
Problems .................................................. 44

6 Concluding remarks ......................................... 45

Acknowledgements ........................................ 45

A Notation and conventions ................................ 46
When I became a student of Pomeranchuk in 1950 I heard from him a kind of joke that the Book of Physics had two volumes: vol.1 is “Pumps and Manometers”, vol.2 is “Quantum Field Theory”

Lev Okun

0 Prologue

The development of Quantum Field Theory is surely one of the most important achievements in modern physics. Presently, all observational evidence points to the fact that Quantum Field Theory (QFT) provides a good description of elementary particles for a wide range of energies, leading up to the Planck mass, $E \lesssim M_{\text{Planck}} \simeq 10^{19}$ GeV. Historically, Quantum Electrodynamics (QED) emerged as the prototype of modern QFT’s. It was developed in the late 1940s and early 1950s chiefly by Feynman, Schwinger and Tomonaga, and is perhaps the most successful theory in physics: the anomalous magnetic dipole moment of the electron is predicted by QED with an accuracy of one part in $10^{10}$.

The scope of these lectures is to provide an introduction to the formalism of Quantum Field Theory. This is best explained by restricting the discussion to the quantum theory of scalar fields. Furthermore, I shall use the Lagrangian formalism and canonical quantisation, thus leaving aside the quantisation approach via path integrals. Since the main motivation for these lectures is the discussion of the underlying formalism leading to the derivation of Feynman rules, the canonical approach is totally adequate. The physically relevant theories of QED, QCD and the electroweak model are covered in the lectures by Nick Evans, Michael Krämer and Adrian Signer.

The outline of these lecture notes is as follows: in section 1 we shall review the two main ingredients of QFT, namely Quantum Mechanics and Special Relativity. Section 2 discusses the relativistic wave equation for free scalar fields, i.e. the Klein-Gordon equation. We shall derive a consistent quantum interpretation of its solutions in terms of the Klein-Gordon field. In section 3 the Lagrangian formalism will be discussed, which is based on the Principle of least Action and provides a general framework to impose the canonical quantisation rules. The more interesting case of interacting scalar fields is presented in section 4: we shall introduce the $S$-matrix and examine its relation with the Green’s functions of the theory. Finally, in section 5 the general method of perturbation theory is presented, which serves to compute the Green functions in terms of a power series in the coupling constant. Here, Wick’s Theorem is of central importance in order to understand the derivation of Feynman rules.

1 Ingredients of Quantum Field Theory

1.1 Basic concepts of Quantum Mechanics

Let us recall that physical states in Quantum Mechanics are described by vectors $|\psi\rangle$ in a Hilbert space $\mathcal{H}$. On the other hand, physical observables are described by hermitian
operators acting on the state vectors $|\psi\rangle$. Common examples for operators are

position operator : $\hat{x}$
momentum operator : $\hat{p} = -i\hbar \nabla$

Hamiltonian : $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) = -\frac{\hbar^2 \nabla^2}{2m} + V(\hat{x})$. \hspace{1cm} (1.1)

The required hermiticity of operators implies that these have real eigenvalues, so that their expectation values can be interpreted as the result of a measurement.

The definition of the Hamiltonian follows the expression for the total energy in classical mechanics, i.e.

$$ T + V = \frac{\hat{p}^2}{2m} + V(x). $$ \hspace{1cm} (1.2)

If the potential energy $V$ vanishes, then the total energy $E$ is given by

$$ E = \frac{\hat{p}^2}{2m} $$ \hspace{1cm} (1.3)

which is the classical, non-relativistic energy-momentum relation. In Quantum Mechanics, the operators must satisfy commutation relations, for instance,

$$ \left[ \hat{x}, \hat{p} \right] = i\hbar $$ \hspace{1cm} (1.4)

$$ \left[ \hat{x}, \hat{x} \right] = \left[ \hat{p}, \hat{p} \right] = 0. $$ \hspace{1cm} (1.5)

Equation (1.4) expresses the well-known fact that precise and independent measurements of $x$ and $p$ cannot be made. To summarise the above, one may say that quantisation is achieved by replacing physical observables by the corresponding hermitian operators and imposing suitable commutation relations.

Quantum Mechanics can be formulated in terms of different so-called quantum pictures, which treat the time dependence of state vectors and operators in different ways. In the Schrödinger picture state vectors are time-dependent, whereas operators describing observables are time-independent. The state vector $|\psi\rangle$ can be represented by the wavefunction $\Psi(x, t)$, which satisfies the Schrödinger equation:

$$ i\hbar \frac{\partial}{\partial t} \Psi(x, t) = \hat{H} \Psi(x, t) $$ \hspace{1cm} (1.6)

The expectation value of some hermitian operator $\hat{O}$ at a given time $t$ is then defined as

$$ \langle \hat{O} \rangle_t = \int d^3x \Psi^\ast(x, t)\hat{O}\Psi(x, t), $$ \hspace{1cm} (1.7)

and the normalisation of the wavefunction is given by

$$ \int d^3x \Psi^\ast(x, t)\Psi(x, t) = \langle 1 \rangle_t. $$ \hspace{1cm} (1.8)

Since $\Psi^\ast\Psi$ is positive, it is natural to interpret it as the probability density for finding a particle at position $x$. Furthermore one can derive a conserved current $\vec{j}$ as well as a continuity equation by considering

$$ \Psi^\ast \times (\text{Schr.Eq.}) - \Psi \times (\text{Schr.Eq.})^\ast. $$ \hspace{1cm} (1.9)
The continuity equation reads
\[ \frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{j} \] (1.10)
where the density \( \rho \) and the current \( \mathbf{j} \) are given by
\[ \rho = \Psi^* \Psi \] (positive), (1.11)
\[ \mathbf{j} = \frac{\hbar}{2im} (\Psi^* \nabla \Psi - (\nabla \Psi^*) \Psi) \] (real). (1.12)

Now that we have derived the continuity equation let us discuss the probability interpretation of Quantum Mechanics in more detail. Consider a finite volume \( V \) with boundary \( S \). The integrated continuity equation is
\[ \int_V \frac{\partial \rho}{\partial t} d^3x = -\int_V \nabla \cdot \mathbf{j} d^3x = -\int_S \mathbf{j} \cdot dS \] (1.13)
where in the last line we have used Gauss’s theorem. Using eq.(1.8) the lhs. can be rewritten and we obtain
\[ \frac{\partial}{\partial t} \langle 1 \rangle_t = -\int_S \mathbf{j} \cdot dS = 0. \] (1.14)
In other words, provided that \( \mathbf{j} = 0 \) everywhere at the boundary \( S \), we find that the time derivative of \( \langle 1 \rangle_t \) vanishes. Since \( \langle 1 \rangle \), represents the total probability for finding the particle anywhere inside the volume \( V \), we conclude that this probability must be conserved; particles cannot be created or destroyed in our theory. Non-relativistic Quantum Mechanics thus provides a consistent formalism to describe a single particle. The quantity \( \Psi(x, t) \) is interpreted as a one-particle wave function.

### 1.2 Special Relativity

Before we try to extend the formalism of Quantum Mechanics to relativistic systems, let us first recall some basic facts about special relativity. Let \( x^\mu = (x^0, \mathbf{x}) \) denote a 4-vector, where \( x^0 = ct \) is the time coordinate (our conventions for 4-vectors, the metric tensor, etc. are listed in Appendix A). Suppose that someone sends out a light signal at time \( y^0 \) and position \( y \). In the time interval \( (x^0 - y^0) \) the light travels a distance \( |x - y| \), so that
\[ (x - y)^2 = (x^0 - y^0)^2 \]
\[ (x^0 - y^0)^2 - (x - y)^2 \equiv (x - y)^2 = 0. \] (1.15)
This defines the light cone about \( y \). In the forward light cone (FLC) the concepts of “before” and “after” make sense. One important postulate of special relativity states that no signal and no interaction can travel faster than the speed of light. This has important consequences about the way in which different events can affect each other. For instance, two events which are characterised by space-time points \( x^\mu \) and \( y^\mu \) are said to be causal if the distance \( (x - y)^2 \) is time-like, i.e. \( (x - y)^2 > 0 \). By contrast, two events characterised by a space-like separation, i.e. \( (x - y)^2 < 0 \), cannot affect each other, since the point \( x \) is not contained inside the light cone about \( y \).
(x - y)^2 > 0, time-like
(x - y)^2 = 0, light-like
(x - y)^2 < 0, space-like

Figure 1: The light cone about \( y \). Events occurring at points \( x \) and \( y \) are said to be time-like (space-like) if \( x \) is inside (outside) the light cone about \( y \).

In non-relativistic Quantum Mechanics the commutation relations among operators indicate whether precise and independent measurements of the corresponding observables can be made. If the commutator does not vanish, then a measurement of one observables affects that of the other. From the above it is then clear that the issue of causality must be incorporated into the commutation relations of the relativistic version of our quantum theory: whether or not independent and precise measurements of two observables can be made depends also on the separation of the 4-vectors characterising the points at which these measurements occur. We will return to this important issue in section 2.

1.3 Relativistic wave equation: The Klein-Gordon equation

In analogy to the non-relativistic case we shall now derive a relativistically invariant wave equation. Our starting point is the relativistic energy-momentum relation

\[ E^2 = p^2 + m^2 \]  

where we have set \( c = 1 \). We may try to quantise the theory by replacing observables by the corresponding hermitian operators, which gives

\[-\hbar^2 \frac{\partial^2}{\partial t^2} \phi(x,t) = -\hbar^2 \nabla^2 \phi(x,t) + m^2 \phi(x,t) , \]  

where \( \phi(x,t) \) is the wavefunction. This can be rewritten as

\[ \hbar^2 \left( \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \phi(x,t) + m^2 \phi(x,t) = 0 \]

\[ \Rightarrow \ (\hbar^2 \Box + m^2) \phi(x,t) = 0 , \]  

which is the Klein-Gordon equation (KGE) for a free particle. As in the non-relativistic case one can derive a continuity equation:

\[ \phi^* \times (\text{KGE}) - \phi \times (\text{KGE})^* . \]  

(1.19)
The density and current in this case are given by
\[
\rho = i \left( \phi^* \frac{\partial}{\partial t} \phi - \frac{\partial}{\partial t} (\phi^*) \phi \right) \quad \text{(real)} \tag{1.20}
\]
\[
\dot{j} = -i (\phi^* \nabla \phi - (\nabla \phi^*) \phi) \quad \text{(real)} \tag{1.21}
\]
However, if we try to take over the probabilistic interpretation encountered in the non-relativistic case, we run into a number of problems:

1. Although \( \int_V \rho \, d^3x \) is again independent of time, the quantity \( \rho \), though real, is not positive definite, which means that \( \rho \) cannot be interpreted as a probability density;

2. One might try to interpret \( \phi^* \phi \) as a probability density instead, but then one finds that \( \phi^* \phi \) is not conserved, since it does not satisfy a continuity equation. In other words, it cannot be shown that \( \int_V \phi^* \phi \, d^3x \) is independent of time;

3. There are states of arbitrarily low energies:
\[
E = \pm \sqrt{p^2 + m^2} \tag{1.22}
\]
so that the system has no ground state.

Therefore we will have to abandon the interpretation which was so successful in the non-relativistic case.

So far we have assumed that the number of particles should be constant in the relativistic theory. If we drop this requirement and introduce the concept of particle creation or annihilation, then the interpretation of \( \phi \) as a one-particle wavefunction does not have to be upheld. As a consequence, \( \int_V \phi^* \phi \, d^3x \) may vary over time due to the change in the number of particles. Although this apparently overcomes the problem that a consistent relativistic theory for one particle cannot be formulated, the problem of negative energies remains unsolved. As we shall see, this is ultimately possible if we re-interpret \( \phi \) as a field operator which can destroy or create particles.

Problems

1.1 Starting from the Schrödinger equation for the wavefunction \( \Psi(x, t) \),
\[
\left\{ -\frac{\hbar^2 \nabla^2}{2m} + V(x) \right\} \Psi(x, t) = i\hbar \frac{\partial}{\partial t} \Psi(x, t)
\]
show that the probability density \( \rho = \Psi^* \Psi \) satisfies the continuity equation eq. (1.10), with the current \( j \) given by eq. (1.12). Verify that the continuity equation can be written in manifestly covariant form, i.e.
\[
\partial_{\mu} j^\mu = 0, \quad j^\mu = (cp, j).
\]

1.2 Derive the corresponding continuity equation for the Klein-Gordon equation using a similar procedure as in Problem 1.1. Set \( c = 1 \) and verify that the continuity equation can be written as \( \partial_{\mu} j^\mu = 0 \) where
\[
\dot{j}_\mu = i \left( \phi^* (\partial_\mu \phi) - (\partial_\mu \phi^*) \phi \right).
\]
Quantisation of the free scalar field

We are now going to discuss the Klein-Gordon equation for free, i.e. non-interacting particles in more detail, finally arriving at a consistent way to quantise it. From now on we will always set $\hbar = c = 1$.

2.1 Heisenberg picture

Before we are ready to discuss solutions of the Klein-Gordon equation we have to look at the equations of motion in more detail, and derive a result which will prove useful for the later analysis. As mentioned earlier, the physical states of a system are represented by vectors in a Hilbert space. We have already encountered the Schrödinger picture, where the state vectors have an explicit time dependence. In order to indicate this we shall use the notation $|t; S\rangle$ for a Schrödinger state at time $t$ which satisfies the Schrödinger equation:

$$i\frac{\partial}{\partial t}|t; S\rangle = \hat{H}|t; S\rangle.$$  \hspace{1cm} (2.1)

By contrast, operators have no explicit time dependence in the Schrödinger picture. The formal solution of eq. (2.1) in a time interval from $t_0$ to $t$ is given by

$$|t; S\rangle = e^{-i\hat{H}(t-t_0)}|t_0; S\rangle,$$  \hspace{1cm} (2.2)

which is easily verified by inserting it into the Schrödinger equation.

Let us consider the expectation value of an observable $\hat{A}$, i.e.

$$\langle t; S|\hat{A}|t; S\rangle \equiv \int d^3x \bar{\Psi}(x) \hat{A} \Psi(x),$$  \hspace{1cm} (2.3)

where the subscript “S” on $\hat{A}$ denotes that the corresponding operator is defined in the Schrödinger picture. It is possible to trade off the time dependence between the state vectors and the operators, since both are merely mathematical devices to describe physical quantities. The expectation value, however, has to remain unchanged. Using the solution eq. (2.2) we obtain

$$\langle t; S|\hat{A}|t; S\rangle = \langle t_0; S|e^{i\hat{H}(t-t_0)} \hat{A} e^{-i\hat{H}(t-t_0)}|t_0; S\rangle.$$  \hspace{1cm} (2.4)

We can thus define the time-dependent operator $\hat{A}_H(t)$ and the time-independent state vector $|H\rangle$ through

$$\hat{A}_H(t) = e^{i\hat{H}(t-t_0)} \hat{A} e^{-i\hat{H}(t-t_0)}$$  \hspace{1cm} (2.5)

$$|H\rangle \equiv |t_0; H\rangle = |t_0; S\rangle.$$  \hspace{1cm} (2.6)

Then we have

$$\langle t; S|\hat{A}_S|t; S\rangle = \langle H|\hat{A}_H(t)|H\rangle.$$  \hspace{1cm} (2.7)

The above relations between $\hat{A}_S$ and $\hat{A}_H(t)$ and the respective state vectors define the Heisenberg picture, in which operators and states are labelled by “H”. The state vectors $|H\rangle$ of the Heisenberg picture are time-independent but may carry a time label, which
simply means that these vectors are identified with the corresponding Schrödinger states at that particular time (see eq. (2.6)). Using eq. (2.5) it is easy to derive

\[ \frac{\partial}{\partial t} \hat{A}_H(t) = i \left[ \hat{H}, \hat{A}_H(t) \right], \tag{2.8} \]

which is the Heisenberg equation of motion.

State vectors in both pictures can depend on a spatial coordinate \( x \). Let us consider a Heisenberg state \( |x; H\rangle \) and compute \( [\hat{p}_j, \hat{x}_k] |x; H\rangle \):

\[ [\hat{p}_j, \hat{x}_k] |x; H\rangle = \hat{p}_j \hat{x}_k |x; H\rangle - \hat{x}_k \hat{p}_j |x; H\rangle \]

\[ = -i \frac{\partial}{\partial x_j} (\hat{x}_k |x; H\rangle) + i x_k \frac{\partial}{\partial x_j} |x; H\rangle \]

\[ = -i \frac{\partial x_k}{\partial x_j} |x; H\rangle - i x_k \frac{\partial}{\partial x_j} |x; H\rangle + i x_k \frac{\partial}{\partial x_j} |x; H\rangle \]

\[ = -i \delta_{jk} |x; H\rangle. \tag{2.9} \]

(The indices \( j, k \) refer to the components of \( x \) and \( p \)). What we have recovered is the uncertainty relation, \([\hat{p}_j, \hat{x}_k] = -i \delta_{jk}\). This result can easily be generalised to Heisenberg operators \( \hat{A}_H \) which have an explicit dependence on \( x \):

\[ [\hat{p}_j, \hat{A}_H(x, t)] = -i \frac{\partial}{\partial x_j} \hat{A}_H(x, t). \tag{2.10} \]

To summarise, any Heisenberg operator satisfies the following equations of motion:

\[ \frac{\partial}{\partial t} \hat{A}_H(x, t) = i \left[ \hat{H}, \hat{A}_H(x, t) \right], \tag{2.11} \]

\[ \frac{\partial}{\partial x_j} \hat{A}_H(x, t) = i \left[ \hat{p}_j, \hat{A}_H(x, t) \right]. \tag{2.12} \]

If, in accordance with the conventions in Appendix A, we use the 4-vectors

\[ \frac{\partial}{\partial x_\mu} = \left( \frac{\partial}{\partial t}, \mathbf{\nabla} \right) \Rightarrow \hat{p}^\mu = \left( \hat{H}, \hat{p} \right) \tag{2.13} \]

then we can rewrite the equations of motion in covariant form

\[ \frac{\partial}{\partial x_\mu} \hat{A}_H(x) = i \left[ \hat{p}^\mu, \hat{A}_H(x) \right], \quad x \equiv x^\mu = (t, x). \tag{2.14} \]

These are called the generalised Heisenberg equations of motion.

### 2.2 Plane wave solutions of the Klein-Gordon equation

In covariant form the Klein-Gordon equation reads

\[ (\Box + m^2) \phi(x) = 0. \tag{2.15} \]

Let us consider real solutions, characterised by \( \phi^*(x) = \phi(x) \) and interpret \( \phi(x) \) as a classical, real field. Here, “classical” simply means “not quantised”, and by using the
concept of “fields” we make the analogy with electromagnetism. It is well known that in Maxwell’s theory the scalar and vector potentials (denoted by $\Phi$ and $\mathbf{V}$, respectively) satisfy wave equations of the type

$$\Box \Phi(x) = 0, \quad \Box V_i(x) = 0, \quad i = 1, 2, 3,$$

and the same is true for the components of the electric and magnetic fields, i.e.

$$\Box E_i(x) = 0, \quad \Box B_i(x) = 0, \quad i = 1, 2, 3.$$  

(2.17)

These wave equations are identical to the Klein-Gordon equation, except for the mass term, which is absent in the Maxwell theory. This illustrates the close relationship of a relativistically invariant theory like classical electromagnetism with the Klein-Gordon equation and motivates the interpretation of $\phi$ as a field variable.

To find the solutions of the Klein-Gordon equation let us consider an ansatz of plane waves

$$\phi(x) \propto e^{i(k^0 t - \mathbf{k} \cdot \mathbf{x}).}$$

(2.18)

The Klein-Gordon equation is satisfied if $(k^0)^2 - \mathbf{k}^2 = m^2$ so that

$$k^0 = \pm \sqrt{\mathbf{k}^2 + m^2}. \quad (2.19)$$

If we choose the positive branch of the square root then we can define the energy as

$$E(k) = \sqrt{\mathbf{k}^2 + m^2} > 0 \quad (2.20)$$

and obtain two types of solutions which read

$$\phi_+(x) \propto e^{i(E(k)t - \mathbf{k} \cdot \mathbf{x})}, \quad \phi_-(x) \propto e^{-i(E(k)t - \mathbf{k} \cdot \mathbf{x})}. \quad (2.21)$$

The general solution is a superposition of $\phi_+$ and $\phi_-$. Using

$$E(k)t - \mathbf{k} \cdot \mathbf{x} = k^\mu k_\mu = k^2 \mathbf{k} \cdot \mathbf{x}$$

(2.22)

this solution reads

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2E(k)} \left( e^{ik \cdot x} \alpha^*(k) + e^{-ik \cdot x} \alpha(k) \right), \quad (2.23)$$

where $\alpha(k)$ is an arbitrary complex coefficient. From the general solution one easily reads off that $\phi$ is real, i.e. $\phi = \phi^*$. 

### 2.3 Quantisation of the real Klein-Gordon field

Starting from the classical solution to the KGE in eq. (2.23) we will now derive a quantum interpretation of $\phi$. Recall that quantisation is achieved by replacing observables by the corresponding operators, as well as the imposition of suitable commutation relations among these operators. In electromagnetism one can indeed express one component of
the electric field, $E_i(x)$, which is a classical observable, by the expectation value of a field operator, i.e.

$$E_i(x) = \langle |\hat{E}_i(x)| \rangle.$$  \hspace{1cm} (2.24)

classical observable exp. value of field operator  \hspace{1cm} (2.25)

Historically this is how a field theory was quantised for the first time. Here we have used empty bra and ket symbols to indicate that the states have to be specified in more detail – this is discussed later. If we follow the same approach for the real Klein-Gordon field, then we have to replace the field $\phi$ by the expectation value of the field operator $\hat{\phi}$:

$$\phi(x) = \langle |\hat{\phi}(x)| \rangle.$$  \hspace{1cm} (2.26)

The condition $\phi = \phi^*$ is obtained if $\hat{\phi}^\dagger = \hat{\phi}$. In other words, $\phi$ is a real solution of the KGE if the operator $\hat{\phi}$ is hermitian and satisfies the KGE as well. Furthermore, $\hat{\phi}$ is a Heisenberg operator and must therefore satisfy the generalised Heisenberg equation (c.f. eq. (2.14)):

$$\frac{\partial}{\partial x_\mu} \phi(x) = i \left[ \hat{p}^\mu, \phi(x) \right].$$  \hspace{1cm} (2.27)

From the classical solution $\phi(x)$ of eq. (2.23) we can deduce the following form of $\hat{\phi}(x)$

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3 2E(k)} \left( e^{ik\cdot x} \hat{a}^\dagger(k) + e^{-ik\cdot x} \hat{a}(k) \right).$$  \hspace{1cm} (2.28)

Here $\hat{a}^\dagger(k)$ and $\hat{a}(k)$ are operators, whose properties can be worked out by inserting the expression for $\phi$ into the generalised Heisenberg equation, which gives

$$\frac{\partial}{\partial x_\mu} \hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3 2E(k)} \left\{ e^{ik\cdot x} (ik^\mu) \hat{a}^\dagger(k) + e^{-ik\cdot x} (-ik^\mu) \hat{a}(k) \right\}$$

$$= \int \frac{d^3k}{(2\pi)^3 2E(k)} \left\{ e^{ik\cdot x} i \left[ \hat{p}^\mu, \hat{a}^\dagger(k) \right] + e^{-ik\cdot x} i \left[ \hat{p}^\mu, \hat{a}(k) \right] \right\}. \hspace{1cm} (2.29)$$

By comparing the integrands we find

$$\left[ \hat{p}^\mu, \hat{a}^\dagger(k) \right] = k^\mu \hat{a}^\dagger(k)$$  \hspace{1cm} (2.30)

$$\left[ \hat{p}^\mu, \hat{a}(k) \right] = -k^\mu \hat{a}(k)$$  \hspace{1cm} (2.31)

These relation will help later us to specify the state vectors in more detail. Let us now consider the vacuum state, which does not contain any particles. This state is denoted by $|0\rangle$, and it we normalise it to one, i.e.

$$\langle 0 | 0 \rangle = 1.$$  \hspace{1cm} (2.32)

Furthermore, the vacuum has zero energy and momentum, which implies that the momentum operator $\hat{p}^\mu$ has a zero eigenvalue when acting on $|0\rangle$:

$$\hat{p}^\mu |0\rangle = 0.$$  \hspace{1cm} (2.33)
Now we consider \( [\hat{p}^\mu, \hat{a}^\dagger(k)]|0\rangle \). Using eq. (2.30) we find that
\[
[\hat{p}^\mu, \hat{a}^\dagger(k)]|0\rangle = \hat{p}^\mu \hat{a}^\dagger(k)|0\rangle - \hat{a}^\dagger(k)\hat{p}^\mu|0\rangle = k^\mu \hat{a}^\dagger(k)|0\rangle.
\]
\[
\Rightarrow \hat{p}^\mu \hat{a}^\dagger(k)|0\rangle = k^\mu \hat{a}^\dagger(k)|0\rangle.
\] (2.34)
In other words, \( \hat{a}^\dagger(k)|0\rangle \) is an eigenstate of \( \hat{p}^\mu \) with eigenvalue \( k^0 = (E(k), k) \), where \( k^\mu \) is the 4-momentum of a relativistic particle with mass \( m \), and \( E(k) = \sqrt{k^2 + m^2} \). Therefore it makes sense to interpret \( \hat{a}^\dagger(k)|0\rangle \) as a one-particle state. Similarly we find that
\[
\hat{p}^\mu \hat{a}(k)|0\rangle = -k^\mu \hat{a}(k)|0\rangle.
\] (2.35)
However, this means that \( \hat{a}(k)|0\rangle \) is a state with negative energy, \( k^0 = -E(k) < 0 \). Such states are clearly unphysical, so that we define
\[
\hat{a}(k)|0\rangle = 0.
\] (2.36)
Furthermore, one can show by applying eqs. (2.30) and (2.31), that
\[
\hat{p}^\mu \hat{a}^\dagger(k_2)\hat{a}^\dagger(k_1)|0\rangle = (k^\mu_1 + k^\mu_2)\hat{a}^\dagger(k_2)\hat{a}^\dagger(k_1)|0\rangle
\] (2.37)
\[
\hat{p}^\mu \hat{a}(k_2)\hat{a}^\dagger(k_1)|0\rangle = (k^\mu_1 - k^\mu_2)\hat{a}(k_2)\hat{a}^\dagger(k_1)|0\rangle.
\] (2.38)
It is thus natural to interpret \( \hat{a}^\dagger(k_2)\hat{a}^\dagger(k_1)|0\rangle \) as a two-particle state, and, after repeating this procedure \( n \) times one obtains
\[
\hat{a}^\dagger(k_n) \cdots \hat{a}^\dagger(k_1)|0\rangle \quad n\text{-particle state}
\] (2.39)
The operators \( \hat{a}^\dagger(k) \) and \( \hat{a}(k) \) increase, respectively decrease the number of particles in a given state by one. We call
\[
\hat{a}^\dagger(k) : \text{creation operator} \quad \hat{a}(k) : \text{annihilation operator}.
\] (2.40)
The states which appear in our quantised version of the Klein-Gordon equation are thus multi-particle states. We still have to look at the normalisation of these states in detail, and also introduce the commutation rules for our field operator \( \hat{\phi} \).

### 2.4 Causality and commutation relations

In section 1 it was mentioned that commutation relations of operators in the relativistic case must be imposed in accordance with the principle of causality, which is central in Special Relativity. Consider two 4-vectors, \( x \) and \( x' \), such that
\[
(x - x')^2 < 0 \quad \text{(space-like)}.
\] (2.41)
In this case \( x \) is outside the light cone about \( x' \) and vice versa. Events occurring at \( x \) and \( x' \) respectively (for instance, measurements of some observable), cannot affect each
other – they are independent. It is then clear that the field operators \( \hat{\phi}(x) \) and \( \hat{\phi}(x') \) must commute

\[
[\hat{\phi}(x), \hat{\phi}(x')] = 0 \quad \text{for} \quad (x - x')^2 < 0.
\] (2.42)

This condition is sometimes called micro-causality. Equation (2.42) is an important relation in order to derive the commutation rules for the creation and annihilation operators \( \hat{a} \) and \( \hat{a}^\dagger \). Our starting point is

\[
\left[ \hat{\phi}(x, t), \hat{\phi}(x', t') \right] = 0, \quad |t' - t| < |x - x'| \neq 0,
\] (2.43)

which is just eq. (2.42) with explicit space and time coordinates. As long as \( |t' - t| < |x - x'| \), the commutator vanishes in a finite interval \( |t' - t| \). It also vanishes for \( t' = t \), hence

\[
\left[ \hat{\phi}(x, t), \hat{\phi}(x', t) \right] = 0, \quad \left[ \hat{\phi}(x, t), \frac{\partial}{\partial t} \hat{\phi}(x', t) \right] = 0, \quad x \neq x'.
\] (2.44)

These expressions are referred to as the equal time commutators of the field operator \( \hat{\phi} \). Before we can use them in order to deduce the commutators for creation and annihilation operators, we rewrite \( \hat{\phi} \), by letting \( k \rightarrow -k \) in the term proportional to \( \hat{a}(k) \) in eq. (2.28).

\[
\hat{\phi}(x, t) = \int \frac{d^3k}{(2\pi)^3} 2E(k) e^{-ik\cdot x} \left\{ e^{iEt} \hat{a}^\dagger(k) + e^{-iEt} \hat{a}(-k) \right\}.
\] (2.45)

Similarly one obtains

\[
\partial_0 \hat{\phi}(x, t) \equiv \frac{\partial}{\partial t} \hat{\phi}(x, t) = \int \frac{d^3k}{(2\pi)^3} i \frac{2}{3} e^{-ik\cdot x} \left\{ e^{iEt} \hat{a}^\dagger(k) - e^{-iEt} \hat{a}(-k) \right\}.
\] (2.46)

The Fourier transform of \( \hat{\phi}(x, t) \) reads

\[
\int d^3x \ e^{ip\cdot x} \hat{\phi}(x, t) = \int \frac{d^3k}{(2\pi)^3} 2E(k) \int d^3x \ e^{i(p-k)\cdot x} \left\{ e^{iEt} \hat{a}^\dagger(k) + e^{-iEt} \hat{a}(-k) \right\}
\]

\[
= \frac{1}{2E} \left\{ e^{iEt} \hat{a}^\dagger(p) + e^{-iEt} \hat{a}(-p) \right\},
\] (2.47)

where \( E = E(p) \) in the last line. Thus we find

\[
2E \int d^3x \ e^{ip\cdot x} \hat{\phi}(x, t) = \left\{ e^{iEt} \hat{a}^\dagger(p) + e^{-iEt} \hat{a}(-p) \right\}.
\] (2.48)

and a similar procedure applied to the Fourier transform of \( \partial_0 \hat{\phi}(x, t) \) yields

\[
-2i \int d^3x \ e^{ip\cdot x} \frac{\partial}{\partial t} \hat{\phi}(x, t) = \left\{ e^{iEt} \hat{a}^\dagger(p) - e^{-iEt} \hat{a}(-p) \right\}.
\] (2.49)

If we take the result of eq. (2.48) we find

\[
\left[ e^{iEt} \hat{a}^\dagger(p_1) + e^{-iEt} \hat{a}(-p_1), e^{iEt} \hat{a}^\dagger(p_2) + e^{-iEt} \hat{a}(-p_2) \right]
\]

\[
= 2E_1 2E_2 \int d^3x \int d^3y \ e^{ip_1\cdot x} e^{ip_2\cdot y} \left[ \hat{\phi}(x, t), \hat{\phi}(y, t) \right].
\] (2.50)
The equal time commutator tells us that the rhs. must vanish, which gives
\[ e^{i(E_1+E_2)t} \left[ \hat{a}^\dagger(p_1), \hat{a}^\dagger(p_2) \right] + e^{-i(E_1-E_2)t} \left[ \hat{a}(-p_1), \hat{a}^\dagger(p_2) \right] \\
+ e^{i(E_1-E_2)t} \left[ \hat{a}^\dagger(p_1), \hat{a}(-p_2) \right] + e^{-i(E_1+E_2)t} \left[ \hat{a}(-p_1), \hat{a}(-p_2) \right] = 0. \] (2.51)

This relation is valid for all times \( t \), and we deduce that
\[ \left[ \hat{a}^\dagger(p_1), \hat{a}^\dagger(p_2) \right] = \left[ \hat{a}(-p_1), \hat{a}(-p_2) \right] = 0, \] (2.52)

which tells us the important result that both creation and annihilation operators commute among themselves!

Let us discuss the consequences by considering two creation operators, \( \hat{a}^\dagger(p_1) \) and \( \hat{a}^\dagger(p_2) \), acting on the vacuum \( |0\rangle \). The fact that creation operators commute implies that
\[ |p_1, p_2 \rangle \equiv \hat{a}^\dagger(p_1) \hat{a}^\dagger(p_2) |0\rangle = \hat{a}^\dagger(p_2) \hat{a}^\dagger(p_1) |0\rangle = |p_2, p_1 \rangle, \] (2.53)

where the positions of the 3-momenta on the far left- and far right-hand sides refer to particle one and two, respectively. We see that, since \( |p_1, p_2 \rangle = |p_2, p_1 \rangle \), the state \( |p_1, p_2 \rangle \) is symmetric if we exchange the two particles. We have thus derived that our particles must be bosons! The quantised Klein-Gordon theory describes relativistic particles which are bosons, i.e. which have integer spin. The fact that we have considered real solutions to the KGE means that we consider particles without charge. An example of a neutral particle with spin zero is the neutral pion, \( \pi^0 \). The formalism that we have developed so far can be used to describe free neutral pions.

We still have to derive the commutator of \( [\hat{a}(p), \hat{a}^\dagger(q)] \). If we solve eq. (2.48) and (2.49) for \( \hat{a} \) and \( \hat{a}^\dagger \) then we find
\[ [\hat{a}(p), \hat{a}^\dagger(q)] = e^{i(E(p)-E(q)t)} \int d^3x \int d^3y e^{-ip \cdot x} e^{iq \cdot y} \\
\times \left\{ -iE(p) \left[ \hat{\phi}(x, t), \partial_0 \hat{\phi}(y, t) \right] + iE(q) \left[ \partial_0 \hat{\phi}(x, t), \hat{\phi}(y, t) \right] \right\}. \] (2.54)

This expression can be simplified using the commutators for the field operators. The equal time commutator tells us that
\[ \left[ \hat{\phi}(x, t), \partial_0 \hat{\phi}(y, t) \right] = 0 \quad \text{for} \quad x \neq y, \] (2.55)
as required by causality. However, so far we have not imposed any commutation rules for the case \( x = y \). Let us try
\[ \left[ \hat{\phi}(x, t), \partial_0 \hat{\phi}(y, t) \right] = i\delta^3(x - y), \] (2.56)
in analogy with Quantum Mechanics, where \( [\hat{x}_j, \hat{p}_k] = i\delta_{jk} \). We then obtain
\[ [\hat{a}(p), \hat{a}^\dagger(q)] = e^{i(E(p)-E(q)t)} \int d^3x \int d^3y e^{-ip \cdot x} e^{iq \cdot y} \\
\times \left\{ E(p) \delta^3(x - y) + E(q) \delta^3(y - x) \right\} \\
= e^{i(E(p)-E(q)t)} \int d^3x e^{-i(p-q) \cdot x} \underbrace{(E(p) + E(q))}_{(2\pi)^3\delta^3(p-q)}. \] (2.57)
Our commutation rule for $\hat{a}$ and $\hat{a}^\dagger$ thus reads

$$[\hat{a}(p), \hat{a}^\dagger(q)] = (2\pi)^3 2E(p)\delta^3(p - q). \quad (2.58)$$

Armed with this result we can now look at the normalisation of our multi-boson states. Let $|k\rangle$ and $|k'\rangle$ be two one-meson states, i.e.

$$|k\rangle = \hat{a}^\dagger(k)|0\rangle, \quad |k'\rangle = \hat{a}^\dagger(k')|0\rangle. \quad (2.59)$$

Then we have

$$\langle k'|k \rangle = \langle 0|\hat{a}(k')\hat{a}^\dagger(k)|0 \rangle = \langle 0|\left[\hat{a}(k'), \hat{a}^\dagger(k)\right]|0 \rangle = (2\pi)^3 2E(k)\delta^3(k' - k) > 0. \quad (2.60)$$

We see that the way we have set up the commutation rules for $\hat{\phi}$ and $\partial_0\hat{\phi}$ in eq. (2.56) ensures that our multi-meson states have positive norm, i.e.

$$\int \frac{d^3k'}{(2\pi)^3 2E(k')} \langle k'|k \rangle = \int d^3k' E(k) E(k') \delta^3(k' - k) = 1, \quad (2.61)$$

and thus our states form indeed a proper Hilbert space.

To conclude this chapter, let us make a brief summary of what we have actually done. A consistent quantisation of the free Klein-Gordon field has been achieved by going through the following steps:

- Abandon the interpretation of $\phi(x, t)$ as a one-particle wave function and think of it as a field variable.
- Replace the classical field $\phi$ by a field operator $\hat{\phi}$.
- Expand $\hat{\phi}$ in terms of creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$.
- Impose the equal time commutators

$$\left[\hat{\phi}(x, t), \partial_0\hat{\phi}(y, t)\right] = i\delta^3(x - y), \quad \left[\hat{\phi}(x, t), \hat{\phi}(y, t)\right] = 0, \quad (2.62)$$

which ensures that the resulting theory obeys causality.

- This implies the following commutation rules for $\hat{a}^\dagger$, $\hat{a}$:

$$[\hat{a}^\dagger(k_1), \hat{a}^\dagger(k_2)] = 0 \quad (2.63)$$
$$[\hat{a}(k_1), \hat{a}(k_2)] = 0 \quad (2.64)$$
$$[\hat{a}(k_1), \hat{a}^\dagger(k_2)] = (2\pi)^3 2E(k_1)\delta^3(k_1 - k_2) \quad (2.65)$$

Relation (2.63) implies that our particles must be bosons. The state vectors are interpreted as multi-boson (or multi-meson) states:

$$|0\rangle : \text{vacuum, } \langle 0|0 \rangle = 1$$
$$\hat{a}^\dagger(k)|0\rangle = |k\rangle : \text{one-meson state, positive energy}$$
$$\hat{a}(k)|0\rangle = 0 : \text{exclude states with negative energy}. \quad (2.66)$$
Furthermore, relation (2.65) implies that one-meson states must have positive norm. The generalisation to \( n \)-meson states is achieved by repeated applications of creation operators, e.g.

\[
\hat{a}^\dagger(k_1) \cdots \hat{a}^\dagger(k_n) |0\rangle = |k_1, \ldots, k_n\rangle. \tag{2.67}
\]

The resulting Hilbert space of multi-particle states is called a Fock space.

Problems

2.1 Let \( \hat{A}_H(t) \) and \( \hat{A}_S \) be operators in the Heisenberg and Schrödinger pictures, respectively. Using the relation

\[
\hat{A}_H(t) = e^{i\hat{H}(t-t_0)} \hat{A}_S e^{-i\hat{H}(t-t_0)}
\]

derive the Heisenberg equation of motion, eq. (2.8).

2.2 Derive the generalised uncertainty relation, eq. (2.10), for a Heisenberg operator \( \hat{A}_H(x, t) \). \textbf{Hint:} Consider \( \hat{p}_j, \hat{A}_H(x, t) \) |\( x; H\rangle \), where |\( x; H\rangle \) is an arbitrary Heisenberg state.

2.3 Given the relativistic invariance of the measure \( d^4k \), show that the integration measure

\[
\frac{d^3k}{(2\pi)^3 2E(k)}
\]

is Lorentz-invariant, provided that \( E(k) = \sqrt{k^2 + m^2} \). \textbf{Hint:} Start from the expression

\[
\frac{d^4k}{(2\pi)^3} \delta(k^2 - m^2) \theta(k_0)
\]

and use

\[
\delta(x^2 - x_0^2) = \frac{1}{2x} \left( \delta(x - x_0) + \delta(x + x_0) \right)
\]

What is the significance of the \( \delta \) and \( \theta \) functions above?

2.4 Verify that eq. (2.23) is indeed a solution of the Klein-Gordon equation.

2.5 Use the commutator relations

\[
[\hat{p}^\mu, \hat{a}^\dagger(k)] = k^\mu \hat{a}^\dagger(k), \quad [\hat{p}^\mu, \hat{a}(k)] = -k^\mu \hat{a}(k)
\]

to derive eqs. (2.37) and (2.38).

2.6 Starting from the expression for \( \partial_0 \hat{\phi}(x, t) \):

\[
\partial_0 \hat{\phi}(x, t) = \int \frac{d^3k}{(2\pi)^3} \frac{i}{2} e^{-ikx} \left( e^{iEt} \hat{a}^\dagger(k) - e^{-iEt} \hat{a}(k) \right)
\]
invert the Fourier transform to obtain
\[ e^{iEt} \hat{a}^\dagger(k) - e^{-iEt} \hat{a}(-k) = -2i \int d^3 x \, e^{ikx} \partial_0 \hat{\phi}(x, t). \]

Use this result together with eq. (2.48) in the lecture notes, i.e.
\[ e^{iEt} \hat{a}^\dagger(k) + e^{-iEt} \hat{a}(-k) = 2E(k) \int d^3 x \, e^{ikx} \hat{\phi}(x, t). \]
to solve for \( \hat{a}^\dagger(k) \) and \( \hat{a}(k) \). Verify that the expression for the commutator \([\hat{a}(p), \hat{a}^\dagger(q)]\)
reads
\[ [\hat{a}(p), \hat{a}^\dagger(q)] = e^{i(E(p)-E(q))t} \int d^3 x \, d^3 y \, e^{-ipy+ipx} \]
\[ \times i \left\{ E(q) \left[ \partial_0 \hat{\phi}(x, t), \hat{\phi}(y, t) \right] - E(p) \left[ \hat{\phi}(x, t), \partial_0 \hat{\phi}(y, t) \right] \right\}. \]

## 3 Lagrangian formalism

We could have derived all results in the previous section in a much more axiomatic fashion, by starting from the Lagrangian of the theory. The Lagrangian formalism is used in all modern theories of matter, most notably in the Standard Model of Elementary Particle Physics, and its extensions. The starting point for any theoretical treatment of fundamental interactions is the relevant Lagrangian, for instance
\[ L_{\text{QED}}, L_{\text{QCD}}, L_{\text{GSW}} \]
for Quantum Electrodynamics, Quantum Chromodynamics and the electroweak (Glashow-Salam-Weinberg) theory, respectively. In this section we will introduce the Lagrangian formalism in the context of classical mechanics, before we move on to classical field theory and then quantise it.

### 3.1 Classical Mechanics

In Classical Mechanics one considers point particles of mass \( m \) at some position \( x \). A typical problem is then to find the trajectory \( x(t) \) of that particle by integrating its equation of motion, which is
\[ m \ddot{x} = F \equiv -\frac{\partial V}{\partial x}, \]
where \( V \) is the potential. This well-known result is Newton’s 2nd law. The Lagrangian formalism provides a generalisation such that the equation of motion can, in fact, be derived from the Principle of least Action.

The Lagrangian for a single classical particle is defined as
\[ L = T - V = \frac{1}{2}m \dot{x}^2 - V(x) \]
\[ = L(x, \dot{x}). \]
The integral
\[ S = \int_{t_1}^{t_2} L(x, \dot{x}) \, dt \] (3.4)
is called the action. The Principle of least Action then states that the motion of the particle (i.e. its trajectory) must be such that the action \( S \) is a minimum. In other words, the requirement that \( S \) be a minimum singles out one particular trajectory \( x(t) \). Let us minimise \( S \) in the standard fashion by considering a variation in the path \( x(t) \):

\[ x(t) \rightarrow x'(t) = x(t) + \delta x(t), \quad \delta x(t) \ll x. \] (3.5)

As a boundary condition we require that the points at the boundaries stay fixed, i.e.

\[ x'(t_1) = x(t_1) \quad x'(t_2) = x(t_2) \quad \Rightarrow \delta x(t_1) = \delta x(t_2) = 0. \] (3.6)

The variation of \( S \) can then be worked out in detail by considering

\[ S + \delta S = \int_{t_1}^{t_2} L(x + \delta x, \dot{x} + \delta \dot{x}) \, dt, \quad \delta \dot{x} = \frac{d}{dt} \delta x. \] (3.7)

The variation \( \delta S \) is obtained from the Taylor expansion of the integrand about \((x, \dot{x})\)

\[ \delta S = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} \right\} \, dt \]
\[ = \left. \frac{\partial L}{\partial x} \delta x \right|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right\} \delta x \, dt \] (3.8)

The first term in the last line vanishes, because of the boundary conditions. If \( S \) is a minimum, then \( \delta S = 0 \), and since the remaining integral in eq.(3.8) must vanish for arbitrary variations \( \delta x \), this is only possible if the integrand itself vanishes. Thus we have derived the Euler-Lagrange equation

\[ \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0. \] (3.9)

If we insert eq.(3.3) into the Euler-Lagrange equation we obtain

\[ \frac{\partial L}{\partial x} = -\frac{\partial V(x)}{\partial x} = F \]
\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{d}{dt} m\ddot{x} = m\ddot{x} \]
\[ \Rightarrow m\ddot{x} = F = -\frac{\partial V}{\partial x} \quad \text{(Newton’s law)} \] (3.10)

So we find that the equation of motion (here: Newton’s 2nd law) is reproduced by the Principle of least Action.

In this example the Lagrangian formalism and Newton’s law are equivalent, and it is a legitimate question to ask what we have gained. The key advantage of the Lagrangian
formalism is that it helps us to study the implications of symmetries for a given theory. We will return to this point below when we discuss classical field theory.

We shall now establish the relation of the Principle of least Action with the Hamiltonian formalism. To this end we define the conjugate momentum $p$ by

$$ p \equiv \frac{\partial L}{\partial \dot{x}} = m \dot{x}, \quad (3.11) $$

and the Hamiltonian $H$ via

$$ H(x, p) \equiv p \dot{x} - L(x, \dot{x}) = m \dot{x}^2 - \frac{1}{2} m \ddot{x}^2 + V(x) = \frac{1}{2} m \dot{x}^2 + V(x) = T + V. \quad (3.12) $$

The Hamiltonian $H(x, p)$ is the total energy of the system; it is a function of the position variable $x$ and the conjugate momentum$^1 p$. It is now easy to derive Hamilton’s equations (see exercise)

$$ \frac{\partial H}{\partial x} = -\dot{p}, \quad \frac{\partial H}{\partial p} = \dot{x}. \quad (3.13) $$

The Euler-Lagrange equations and Hamilton’s equations provide an entirely equivalent description of the system. We can now have a look at conservation laws. Assuming that the Hamiltonian has no explicit time dependence we find

$$ \frac{d}{dt} H(x, p) = \frac{\partial H}{\partial x} \frac{dx}{dt} + \frac{\partial H}{\partial p} \frac{dp}{dt} = -\dot{p} \dot{x} + \dot{x} \dot{p} = 0, \quad (3.14) $$

and hence we conclude that the total energy $T + V$ is conserved.

### 3.2 Classical field theory

We can extend the Lagrangian formalism for a classical point particle to field theory. The rôle of the particle’s trajectory $x(t)$ is now played by the classical field $\phi(x, t) = \phi(x)$. The “dictionary” which provides the relation to field theory reads as follows:

<table>
<thead>
<tr>
<th>Classical Mechanics:</th>
<th>Classical field theory:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>$x^\mu$</td>
</tr>
<tr>
<td>$x(t)$</td>
<td>$\phi(x)$ (field configuration)</td>
</tr>
<tr>
<td>$L(x, \dot{x})$</td>
<td>$L(\phi, \partial_\mu \phi)$ (Lagrangian density)</td>
</tr>
</tbody>
</table>

(3.15)

The action is defined as

$$ S = \int d^4 x \mathcal{L}(\phi, \partial_\mu \phi), \quad (3.16) $$

where $\mathcal{L}$ is usually referred to as the Lagrangian density, and the Lagrangian is given by

$$ L(\phi, \partial_\mu \phi) = \int d^3 x \mathcal{L}(\phi, \partial_\mu \phi). \quad (3.17) $$

$^1$ It should be noted that the conjugate momentum is in general not equal to $m \dot{x}$.
In order to derive the Euler-Lagrange equations in this case, let us consider variations of the field and its derivative according to

\[ \phi \to \phi + \delta \phi, \quad \partial_\mu \phi \to \partial_\mu \phi + \delta \partial_\mu \phi, \quad \delta \partial_\mu \phi = \partial_\mu \delta \phi. \quad (3.18) \]

The variation of the action becomes

\[ \delta S = \int d^4x \left\{ \frac{\partial L}{\partial \phi} \delta \phi + \frac{\partial L}{\partial (\partial_\mu \phi)} \delta (\partial_\mu \phi) \right\} \]

\[ = \frac{\partial L}{\partial (\partial_\mu \phi)} \delta \phi + \int d^4x \left\{ \frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \phi)} \right\} \delta \phi. \quad (3.19) \]

As in the previous subsection one can argue that the integrand itself must vanish if \( \delta S = 0 \). This yields the Euler-Lagrange equations for a classical field theory:

\[ \frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \phi)} = 0, \quad (3.20) \]

where in the second term a summation over the Lorentz index \( \mu \) is implied.

Let us now consider the Lagrangian

\[ L = \frac{1}{2} \partial_\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 \]

(3.21)

The functional derivatives yield

\[ \frac{\partial L}{\partial \phi} = -m^2 \phi, \quad \frac{\partial L}{\partial (\partial_\mu \phi)} = \partial_\mu \phi \]

(3.22)

so that

\[ \partial_\mu \frac{\partial L}{\partial (\partial_\mu \phi)} = \partial_\mu \partial_\nu \phi = \Box \phi. \quad (3.23) \]

The Euler-Lagrange equation then implies

\[ (\Box + m^2)\phi(x) = 0, \quad (3.24) \]

and one recovers the Klein-Gordon equation from a Lagrangian via the Euler-Lagrange equation. In analogy to classical mechanics we can define a conjugate momentum \( \pi \) through

\[ \pi(x) \equiv \frac{\partial L(\phi, \partial_\mu \phi)}{\partial \phi(x)} = \frac{\partial L(\phi, \partial_\mu \phi)}{\partial (\partial_\nu \phi(x))} = \partial_\nu \phi(x). \quad (3.25) \]

Note that the momentum variables \( p, p_\mu \) and the conjugate momentum \( \pi \) are not the same. The word “momentum” is used only as a semantic analogy to classical mechanics.

In classical mechanics we have shown that the total energy is conserved. If we want to do the same for classical field theory, then we first have to have a look at what is known as Noether’s Theorem. In a nutshell, Noether’s theorem says that the invariance of a Lagrangian under a symmetry transformation implies the existence of a conserved
quantity. For instance, the conservation of 3-momentum $p$ is associated with translational invariance of the Lagrangian, i.e. the transformation

$$ x \to x + a, \quad a: \text{constant 3-vector}, \quad (3.26) $$

while the conservation of energy comes from the invariance of the Lagrangian under time translations

$$ t \to t + \tau, \quad \tau: \text{constant time interval}. \quad (3.27) $$

For our classical field theory one can use Noether’s theorem to derive the following relation

$$ \partial^\mu \left\{ \frac{\partial L}{\partial (\partial^\mu \phi)} \partial_\nu \phi - g_{\mu\nu} L \right\} = 0, \quad (3.28) $$

where the expression in curly brackets is called the energy-momentum tensor, $\Theta_{\mu\nu}$. Equation (3.28) states that the energy-momentum tensor is conserved for every component $\nu$:

$$ \partial^\mu \Theta_{\mu\nu} \equiv \partial^0 \Theta_{0\nu} - \partial^j \Theta_{j\nu} = 0. \quad (3.29) $$

Let us examine $\Theta_{00}$ in more detail:

$$ \Theta_{00} = \frac{\partial L}{\partial (\partial^0 \phi)} \partial_0 \phi - g_{00} L = \pi(x)(\partial_0 \phi(x)) - L. \quad (3.30) $$

The last line is reminiscent of the definition of the Hamiltonian in eq.(3.14), and so we can define the Hamiltonian density $\mathcal{H}$ as

$$ \mathcal{H}(\pi, \phi) \equiv \Theta_{00} = \pi(x)(\partial_0 \phi(x)) - L(\phi, \partial_\mu \phi). \quad (3.31) $$

The conservation of energy can now be shown by considering

$$ \frac{\partial}{\partial t} \int_V d^3 x \Theta_{00} = \int_V d^3 x \partial^0 \Theta_{00} = \int_V d^3 x \partial^j \Theta_{j0} = \int_S dS \cdot \Theta_0 = 0, \quad (3.32) $$

where we have used eq.(3.29) in the second line. The Hamiltonian density is a conserved quantity, provided that there is no energy flow through the surface $S$ which encloses the volume $V$. In a similar manner one can show that 3-momentum $p_j$, which is related to $\Theta_{0j}$, is conserved as well.

We have thus established the Lagrange-Hamilton formalism for classical field theory: we derived the equation of motion (Euler-Lagrange equation) from the Lagrangian and introduced the conjugate momentum. We then defined the Hamiltonian (density) and considered conservation laws by studying the energy-momentum tensor $\Theta_{\mu\nu}$. 

21
3.3 Quantum field theory

We shall now quantise the theory by promoting the classical fields \( \phi, \pi \) to field operators \( \hat{\phi}(x) \) and \( \hat{\pi}(x) \), and by imposing the (equal time) commutators

\[
\begin{align*}
[\hat{\phi}(x,t), \hat{\pi}(y,t)] &= i\delta^3(x-y), \quad \hat{\pi}(x) \equiv \partial_0 \hat{\phi}(x) \quad (3.33) \\
[\hat{\phi}(x,t), \hat{\phi}(y,t)] &= [\hat{\pi}(x,t), \hat{\pi}(y,t)] = 0. \quad (3.34)
\end{align*}
\]

Both operators \( \hat{\phi} \) and \( \hat{\pi} \) can be expanded in terms of creation and annihilation operators \( \hat{a}^\dagger \) and \( \hat{a} \). Let’s work out the Hamiltonian, i.e. the operator of the total energy. The starting point is the expression for the Hamiltonian density

\[ \mathcal{H} = \hat{\pi}(x)(\partial_0 \hat{\phi}(x)) - \mathcal{L}(\hat{\phi}, \hat{\pi}). \quad (3.35) \]

In terms of \( \hat{a}^\dagger \) and \( \hat{a} \) one obtains

\[
\hat{H} \equiv \int d^3x \mathcal{H} = \frac{1}{4} \int \frac{d^3p}{(2\pi)^3} \left( \hat{a}^\dagger(p)\hat{a}(p) + \hat{a}(p)\hat{a}^\dagger(p) \right)
\]

\[
= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3 2E(p)} E(p) \left( \hat{a}^\dagger(p)\hat{a}(p) + \hat{a}(p)\hat{a}^\dagger(p) \right). \quad (3.36)
\]

Let us now work out the energies of some of our multi-particle states. Starting with the vacuum as the ground state we get

\[
\langle 0 | \hat{H} | 0 \rangle = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3 2E(p)} E(p) \left\{ \langle 0 | \hat{a}^\dagger(p)\hat{a}(p) | 0 \rangle + \langle 0 | \hat{a}(p)\hat{a}^\dagger(p) | 0 \rangle \right\}. \quad (3.37)
\]

The first term in curly brackets vanishes, since \( \hat{a} \) annihilates the vacuum. The second can be rewritten as

\[
\hat{a}(p)\hat{a}^\dagger(p) | 0 \rangle = \left\{ \left[ \hat{a}(p), \hat{a}^\dagger(p) \right] + \hat{a}^\dagger(p)\hat{a}(p) \right\} | 0 \rangle. \quad (3.38)
\]

It is now the second term which vanishes, whereas the first can be replaced by the commutator. Thus we obtain

\[
\langle 0 | \hat{H} | 0 \rangle = \delta^3(0) \frac{1}{2} \int d^3p E(p) = \infty, \quad (3.39)
\]

which means that the energy of the ground state is infinite! This result seems rather paradoxical, but it is quite easy to work around it. What we are actually interested in is the energy of multi-particle states \textit{relative} to the vacuum, i.e. the ground state. In this case it does not really matter what the absolute value of the ground state energy is. We can simply redefine the vacuum energy by setting it to zero, thereby subtracting the vacuum contribution from expectation values of the Hamiltonian \( \hat{H} \). Let us make this more explicit by considering the modified Hamiltonian \( \hat{H}' \)

\[
\hat{H}' = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3 2E(p)} E(p) \left\{ \hat{a}^\dagger(p)\hat{a}(p) + \hat{a}(p)\hat{a}^\dagger(p) - \langle 0 | \hat{a}^\dagger(p)\hat{a}(p) + \hat{a}(p)\hat{a}^\dagger(p) | 0 \rangle \right\}
\]

\[
= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3 2E(p)} E(p) \left\{ 2\hat{a}^\dagger(p)\hat{a}(p) + [\hat{a}(p), \hat{a}^\dagger(p)] - \langle 0 | [\hat{a}(p), \hat{a}^\dagger(p)] | 0 \rangle \right\}. \quad (3.40)
\]
Here the subtraction of the vacuum energy is shown explicitly, and we can rewrite is as

\[
\hat{H'} = 1/2 \int \frac{d^3p}{(2\pi)^3 2E(p)} E(p) \hat{a}^\dagger(p) \hat{a}(p) + \frac{1}{2} \int \frac{d^3p}{(2\pi)^3 2E(p)} E(p) \left\{ [\hat{a}(p), \hat{a}^\dagger(p)] - \langle 0 | [\hat{a}(p), \hat{a}^\dagger(p)] | 0 \rangle \right\}.
\]

\[
= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3 2E(p)} E(p) \hat{a}^\dagger(p) \hat{a}(p) + \hat{H}^{\text{vac}} \tag{3.41}
\]

The operator \(\hat{H}^{\text{vac}}\) ensures that the vacuum energy is properly subtracted: if \(|\Psi\rangle\) and \(|\Psi'\rangle\) denote arbitrary Fock states, then it is easy to see that \(\langle \Psi'|\hat{H}^{\text{vac}}|\Psi\rangle = 0\). In particular we now find that

\[
\langle 0 | \hat{H}' | 0 \rangle = 0, \tag{3.42}
\]

as it should be. A simple way to remove the vacuum contribution is to introduce normal ordering. Normal ordering means that all annihilation operators appear to the right of any creation operator. The notation is

\[
: \hat{a} \hat{a}^\dagger : = \hat{a}^\dagger \hat{a}, \tag{3.43}
\]

i.e. the normal-ordered operators are enclosed within colons. For instance

\[
: \frac{1}{2} (\hat{a}^\dagger(p) \hat{a}(p) + \hat{a}(p) \hat{a}^\dagger(p)) : = \hat{a}^\dagger(p) \hat{a}(p). \tag{3.44}
\]

It is important to keep in mind that \(\hat{a}\) and \(\hat{a}^\dagger\) always commute inside \(\cdots\). This is true for an arbitrary string of \(\hat{a}\) and \(\hat{a}^\dagger\). With this definition we can write the normal-ordered Hamiltonian as

\[
: \hat{H} : = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3 2E(p)} E(p) (\hat{a}^\dagger(p) \hat{a}(p) + \hat{a}(p) \hat{a}^\dagger(p)) : \equiv \int \frac{d^3p}{(2\pi)^3 2E(p)} E(p) \hat{a}^\dagger(p) \hat{a}(p). \tag{3.45}
\]

Hence, we find that

\[
\langle \Psi'| : \hat{H} : | \Psi \rangle = \langle \Psi'| \hat{H}' | \Psi \rangle, \tag{3.46}
\]

and, in particular, \(\langle 0 | : \hat{H} : | 0 \rangle = 0\). The normal ordered Hamiltonian thus produces a sensible result for the vacuum energy.

Let us now look at the interpretation of the Hamiltonian in more detail. In particular, we shall be interested in the combination \(\hat{a}^\dagger \hat{a}\), which appears in its definition. Consider a one-meson state, \(|k\rangle\), of a particle with mass \(m\) and momentum \(\vec{k}\). Let us see what happens when the operator \(\int \frac{d^3p}{(2\pi)^3 2E(p)} \hat{a}^\dagger(p) \hat{a}(p)\) acts on \(|k\rangle\):

\[
\int \frac{d^3p}{(2\pi)^3 2E(p)} \hat{a}^\dagger(p) \hat{a}(p) |k\rangle = \int \frac{d^3p}{(2\pi)^3 2E(p)} \hat{a}^\dagger(p) \hat{a}(p) \hat{a}^\dagger(k) |0\rangle
\]

\[
= \int \frac{d^3p}{(2\pi)^3 2E(p)} \hat{a}^\dagger(p) \left\{ [\hat{a}(p), \hat{a}^\dagger(k)] + \hat{a}^\dagger(k) \hat{a}(p) \right\} |0\rangle
\]

\[
= \int d^3p \hat{a}^\dagger(p) \delta^3(p - k) |0\rangle = \hat{a}^\dagger(k) |0\rangle = |k\rangle. \tag{3.47}
\]
We find that $|k\rangle$ is an eigenstate of $\int \frac{d^3p}{(2\pi)^3 2E(p)} \hat{a}^\dagger(p)\hat{a}(p)$ with eigenvalue 1. Let us now look at a two meson state $|\{k,k\rangle$. Using similar manipulations as above we find that

$$\int \frac{d^3p}{(2\pi)^3 2E(p)} \hat{a}^\dagger(p)\hat{a}(p) |\{k,k\rangle = 2 |\{k,k\rangle. \quad (3.48)$$

At this point the pattern becomes clear: the operator

$$\int \frac{d^3p}{(2\pi)^3 2E(p)} \hat{a}^\dagger(p)\hat{a}(p)$$

returns the number of mesons in a given Fock state, and is therefore referred to as the number operator. This is easily generalised to $n$-meson states, i.e. one can derive

$$\int \frac{d^3p}{(2\pi)^3 2E(p)} \hat{a}^\dagger(p)\hat{a}(p) |\{k_1,\ldots,k_n\rangle \quad (3.50)$$

Note that the normal-ordered Hamiltonian differs from the number operator just by a factor of $E(p)$. If we work out the energy of an $n$-meson state we get

$$\hat{H} : |\{k_1,\ldots,k_n\rangle = \int \frac{d^3p}{(2\pi)^3 2E(p)} E(p)\hat{a}^\dagger(p)\hat{a}(p) |\{k_1,\ldots,k_n\rangle = nE(k) |\{k_1,\ldots,k_n\rangle, \quad (3.51)$$

where $E(k) = \sqrt{k^2 + m^2} > 0$. This implies that the energy of the $n$-meson state is positive. One can easily generalise this to a system of $n$ mesons of equal mass but unequal momenta $k_1,\ldots,k_n$:

$$\hat{H} : |\{k_1,\ldots,k_n\rangle = \{E(k_1) + \ldots + E(k_n)\} |\{k_1,\ldots,k_n\rangle. \quad (3.52)$$

We conclude that the Hamiltonian $\hat{H}$ returns the total energy of an $n$-meson state. Since the number operator has strictly non-negative eigenvalues, and since $E(k) > 0$, the energy of any Fock state cannot be negative. This is finally the solution of the negative energy problem encountered earlier.

### 3.4 Summary: canonical quantisation for real scalar fields

1. Start from the classical Lagrangian density

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2} (\partial_{\mu} \phi \partial^\mu \phi) - \frac{1}{2} m^2 \phi^2, \quad \phi(x) : \text{classical field} \quad (3.53)$$

2. The Principle of least Action implies the equation of motion

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_{\mu} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} = 0, \quad \Rightarrow \quad (\Box + m^2) \phi(x) = 0. \quad (3.54)$$

3. Define a conjugate momentum $\pi(x)$ and the Hamiltonian density by

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi(x))} = \partial_0 \phi(x),$$

$$\mathcal{H} = \pi(x)(\partial_0 \phi(x)) - \mathcal{L}(\phi, \partial_\mu \phi) \quad (3.55)$$
4. Quantise the theory by regarding $\phi(x)$, $\pi(x)$ as field operators with equal time commutators

$$\left[ \hat{\phi}(x, t), \hat{\pi}(y, t) \right] = i\delta^3(x - y)$$

$$\left[ \hat{\phi}(x, t), \hat{\phi}(y, t) \right] = \left[ \hat{\pi}(x, t), \hat{\pi}(y, t) \right] = 0.$$  \hspace{1cm} (3.56)

The operators $\hat{\phi}$ and $\hat{\pi}$ are expanded in terms of $\hat{a}^\dagger$ and $\hat{a}$, which are used to generate a basis of Fock states.

5. The energy of a given Fock state is given by the normal-ordered Hamiltonian,

$$: \hat{H} : = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \hat{a}^\dagger(p)\hat{a}(p),$$  \hspace{1cm} (3.57)

acting on that state. Normal ordering ensures that the vacuum energy is subtracted and that the total energy of a given Fock state is non-negative.

As a final remark in this section, let us note that the term “canonical quantisation” refers to the imposition of commutation relations among field operators. An alternative method of quantisation, which is actually more flexible, is based on the path integral formulation. This approach is widely used for the quantisation of non-Abelian gauge theories, but a detailed discussion is beyond the scope of these lectures. More details can be found in standard textbooks on Quantum Field Theory.

Problems

3.1 Starting from the definition of the Hamiltonian, $H(x, p) \equiv p\dot{x} - L(x, \dot{x})$, derive Hamilton’s equations

$$\frac{\partial H}{\partial x} = -\dot{p}, \quad \frac{\partial H}{\partial p} = \dot{x}$$

3.2 Show that the Hamiltonian density $\mathcal{H}$ for a free scalar field is given by

$$\mathcal{H} = \frac{1}{2} \left\{ (\partial_0 \phi)^2 + (\nabla \phi)^2 + m^2 \phi^2 \right\}.$$  Use this result to express the Hamiltonian

$$\hat{H} = \frac{1}{2} \int d^3x \left\{ \partial_0 \hat{\phi}^2 + (\nabla \hat{\phi})^2 + m^2 \hat{\phi}^2 \right\}$$

of the quantised theory in terms of creation and annihilation operators and show that it is given by

$$\hat{H} = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3 2E(p)} E(p) \left\{ \hat{a}^\dagger(p)\hat{a}(p) + \hat{a}(p)\hat{a}^\dagger(p) \right\}$$

3.3 Verify eq. (3.50).
3.4 Using the expressions for $\hat{\phi}$ and $\hat{\pi}$ in terms of $\hat{a}$ and $\hat{a}^\dagger$, show that the unequal time commutator $[\hat{\phi}(x), \hat{\pi}(x')]$ is given by

$$
[\hat{\phi}(x), \hat{\pi}(x')] = \frac{i}{2} \int \frac{d^3p}{(2\pi)^3} \left( e^{ip(x-x')} + e^{-ip(x-x')} \right)
$$

Show that for $t = t'$ one recovers the equal time commutator

$$
[\hat{\phi}(x, t), \hat{\pi}(x', t)] = i\delta^3(x - x')
$$

4 Interacting scalar fields

From now on we shall always discuss quantised real scalar fields. It is then convenient to drop the “hats” on the operators that we have considered up to now.

So far we have only discussed free fields without any interaction between them. As this does not make for a very interesting theory, let us now add an interaction Lagrangian $L_{\text{int}}$. The full Lagrangian $L$ is given by

$$
L = L_0 + L_{\text{int}} \quad (4.1)
$$

where

$$
L_0 = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \quad (4.2)
$$

is the free Lagrangian density discussed before. The Hamiltonian density of the interaction is related to $L_{\text{int}}$ simply by

$$
\mathcal{H}_{\text{int}} = -L_{\text{int}} \quad (4.3)
$$

which follows from its definition. We shall leave the details of $L_{\text{int}}$ unspecified for the moment. What we will be concerned with mostly are scattering processes, in which two initial particles with momenta $p_1$ and $p_2$ scatter, thereby producing a number of particles in the final state, characterised by momenta $k_1, \ldots, k_n$. This is schematically shown in Fig. 2. Our task is to find a description of such a scattering process in terms of the underlying quantum field theory.

![Figure 2](image_url)

**Figure 2**: Scattering of two initial particles with momenta $p_1$ and $p_2$ into $n$ particles with momenta $k_1, \ldots, k_n$ in the final state.
4.1 The $S$-matrix

The timescales over which interactions happen are extremely short. The scattering (interaction) process takes place during a short interval around some particular time $t$ with $-\infty \ll t \ll \infty$. Long before $t$, the incoming particles evolve independently and freely. They are described by a field operator $\phi_{\text{in}}$ defined through

$$\lim_{t \to -\infty} \phi(x) = \phi_{\text{in}}(x). \quad (4.4)$$

Long after the collision the particles in the final state evolve again like in the free theory, and the corresponding operator is

$$\lim_{t \to +\infty} \phi(x) = \phi_{\text{out}}(x). \quad (4.5)$$

The fields $\phi_{\text{in}}$, $\phi_{\text{out}}$ are the asymptotic limits of the Heisenberg operator $\phi$. They both satisfy the free Klein-Gordon equation, i.e.

$$(\Box + m^2)\phi_{\text{in}}(x) = 0, \quad (\Box + m^2)\phi_{\text{out}}(x) = 0. \quad (4.6)$$

Operators describing free fields can be expressed as a superposition of plane waves (see eq. (2.28)). Thus, for $\phi_{\text{in}}$ we have

$$\phi_{\text{in}}(x) = \int \frac{d^3k}{(2\pi)^3 2E(k)} \left( e^{ik \cdot x} \hat{a}_{\text{in}}^\dagger(k) + e^{-ik \cdot x} \hat{a}_{\text{in}}(k) \right), \quad (4.7)$$

with an entirely analogous expression for $\phi_{\text{out}}(x)$. Note that the operators $\hat{a}^\dagger$ and $\hat{a}$ also carry subscripts “in” and “out”.

We can now use the creation operators $\hat{a}_{\text{in}}^\dagger$ and $\hat{a}_{\text{out}}^\dagger$ to build up Fock states from the vacuum. For instance

$$\hat{a}_{\text{in}}^\dagger(p_1) \hat{a}_{\text{in}}^\dagger(p_2) |0\rangle = |p_1, p_2; \text{in}\rangle, \quad (4.8)$$

$$\hat{a}_{\text{out}}^\dagger(k_1) \cdots \hat{a}_{\text{out}}^\dagger(k_n) |0\rangle = |k_1, \ldots, k_n; \text{out}\rangle. \quad (4.9)$$

We must now distinguish between Fock states generated by $\hat{a}_{\text{in}}^\dagger$ and $\hat{a}_{\text{out}}^\dagger$, and therefore we have labelled the Fock states accordingly. In eqs. (4.8) and (4.9) we have assumed that there is a stable and unique vacuum state:

$$|0\rangle = |0; \text{in}\rangle = |0; \text{out}\rangle. \quad (4.10)$$

Mathematically speaking, the $\hat{a}_{\text{in}}^\dagger$’s and $\hat{a}_{\text{out}}^\dagger$’s generate two different bases of the Fock space. Since the physics that we want to describe must be independent of the choice of basis, expectation values expressed in terms of “in” and “out” operators and states must satisfy

$$\langle \text{in} | \phi_{\text{in}}(x) | \text{in} \rangle = \langle \text{out} | \phi_{\text{out}}(x) | \text{out} \rangle. \quad (4.11)$$

Here $|\text{in}\rangle$ and $|\text{out}\rangle$ denote generic “in” and “out” states. We can relate the two bases by introducing a unitary operator $S$ such that

$$\phi_{\text{in}}(x) = S \phi_{\text{out}}(x) S^\dagger \quad (4.12)$$

$$|\text{in}\rangle = S |\text{out}\rangle, \quad |\text{out}\rangle = S^\dagger |\text{in}\rangle, \quad S^\dagger S = 1. \quad (4.13)$$
$S$ is called the _S-matrix_ or _S-operator_. Note that the plane wave solutions of $\phi_{\text{in}}$ and $\phi_{\text{out}}$ also imply that

$$\hat{a}_{\text{in}}^\dagger = S \hat{a}_{\text{out}}^\dagger S^\dagger, \quad \hat{a}_{\text{in}} = S \hat{a}_{\text{out}} S^\dagger.$$  \hspace{1cm} (4.14)

By comparing "in" with "out" states one can extract information about the interaction – this is the very essence of detector experiments, where one tries to infer the nature of the interaction by studying the products of the scattering of particles that have been collided with known energies. As we will see below, this information is contained in the elements of the _S-matrix_.

By contrast, in the absence of any interaction, i.e. for $\mathcal{L}_{\text{int}} = 0$ the distinction between $\phi_{\text{in}}$ and $\phi_{\text{out}}$ is not necessary. They can thus be identified, and then the relation between different bases of the Fock space becomes trivial, $S = 1$, as one would expect.

What we are ultimately interested in are transition amplitudes between an initial state $i$ of, say, two particles of momenta $p_1, p_2$, and a final state $f$, for instance $n$ particles of unequal momenta. Obviously the initial state is characterised by an "in" state, whereas the final one must be an "out" state. The transition amplitude is then given by

$$\langle f, \text{out} | i, \text{in} \rangle = \langle f, \text{out} | S | i, \text{in} \rangle = \langle f, \text{in} | S | i, \text{in} \rangle \equiv S_{fi}.$$ \hspace{1cm} (4.15)

The _$S$-matrix_ element $S_{fi}$ therefore describes the transition amplitude for the scattering process in question. The scattering cross section, which is a measurable quantity is then proportional to $|S_{fi}|^2$. All information about the scattering is thus encoded in the _S-matrix_, which must therefore be closely related to the interaction Hamiltonian density $\mathcal{H}_{\text{int}}$. However, before we try to derive the relation between $S$ and $\mathcal{H}_{\text{int}}$ we have to take a slight detour.

### 4.2 More on time evolution: Dirac picture

The operators $\phi(x, t)$ and $\pi(x, t)$ which we have encountered are Heisenberg fields and thus time-dependent. The state vectors are time-independent in the sense that they do not satisfy a non-trivial equation of motion. Nevertheless, state vectors in the Heisenberg picture can carry a time label. For instance, the "in"-states of the previous subsection are defined at $t = 1$. The relation of the Heisenberg operator $\phi_H(x)$ with its counterpart $\phi_S$ in the Schrödinger picture is given by

$$\phi_H(x, t) = e^{iHt} \phi_S e^{-iHt}, \quad H = H_0 + H_{\text{int}},$$ \hspace{1cm} (4.16)

Note that this relation involves the _full_ Hamiltonian $H = H_0 + H_{\text{int}}$ in the interacting theory. We have already found solutions to the Klein-Gordon equation in the free theory, and so we know how to handle time evolution in this case. It now turns out to be useful to introduce a new quantum picture for the interacting theory, in which the time dependence is governed by $H_0$ only. This is the so-called _Dirac_ or _Interaction_ picture. The relation between fields in the Interaction picture, $\phi_I$, and in the Schrödinger picture, $\phi_S$, is given by

$$\phi_I(x, t) = e^{iH_0t} \phi_S e^{-iH_0t}.$$ \hspace{1cm} (4.17)

At $t = -\infty$ the interaction vanishes, i.e. $H_{\text{int}} = 0$, and hence the fields in the Interaction and Heisenberg pictures are identical, i.e. $\phi_H(x, t) = \phi_I(x, t)$ for $t \to -\infty$. The relation
between $\phi_H$ and $\phi_I$ can be worked out easily:

$$\phi_H(\vec{x}, t) = e^{iHt} \phi_S e^{-iHt}$$

$$= e^{iHt} e^{-iH_0t} \phi_S e^{iH_0t} e^{iHt} e^{-iH_0t}$$

$$= U^{-1}(t) \phi_I(\vec{x}, t) U(t),$$

(4.18)

where we have introduced the unitary operator $U(t)$

$$U(t) = e^{iH_0t} e^{-iHt}, \quad U^\dagger U = 1. \quad (4.19)$$

The field $\phi_H(\vec{x}, t)$ contains the information about the interaction, since it evolves over time with the full Hamiltonian. In order to describe the “in” and “out” field operators, we can now make the following identifications:

$$t \to -\infty : \phi_{\text{in}}(\vec{x}, t) = \phi_I(\vec{x}, t) \equiv \phi_H(\vec{x}, t), \quad (4.20)$$

$$t \to +\infty : \phi_{\text{out}}(\vec{x}, t) = \phi_H(\vec{x}, t). \quad (4.21)$$

Furthermore, since the fields $\phi_I$ evolve over time with the free Hamiltonian $H_0$, they always act in the basis of “in” vectors, such that

$$\phi_{\text{in}}(\vec{x}, t) = \phi_I(\vec{x}, t), \quad -\infty < t < \infty. \quad (4.22)$$

The relation between $\phi_I$ and $\phi_H$ at any time $t$ is given by

$$\phi_I(\vec{x}, t) = U(t) \phi_H(\vec{x}, t) U^{-1}(t). \quad (4.23)$$

As $t \to \infty$ the identifications of eqs. (4.21) and (4.22) yield

$$\phi_{\text{in}} = U(\infty) \phi_{\text{out}} U^\dagger(\infty). \quad (4.24)$$

From the definition of the $S$-matrix, eq. (4.12) we then read off that

$$\lim_{t \to \infty} U(t) = S. \quad (4.25)$$

We have thus derived a formal expression for the $S$-matrix in terms of the operator $U(t)$, which tells us how operators and state vectors deviate from the free theory at time $t$, measured relative to $t_0 = -\infty$, i.e. long before the interaction process.

An important boundary condition for $U(t)$ is

$$\lim_{t \to -\infty} U(t) = 1. \quad (4.26)$$

What we mean here is the following: the operator $U$ actually describes the evolution relative to some initial time $t_0$, which we will normally suppress, i.e. we write $U(t)$ instead of $U(t, t_0)$. We regard $t_0$ merely as a time label and fix it at $-\infty$, where the interaction vanishes. Equation (4.26) then simply states that $U$ becomes unity as $t \to t_0$, which means that in this limit there is no distinction between Heisenberg and Dirac fields.
Using the definition of $U(t)$, eq. (4.19), it is an easy exercise to derive the equation of motion for $U(t)$:

$$i \frac{d}{dt} U(t) = H_{\text{int}}(t) U(t), \quad H_{\text{int}}(t) = e^{iH_0 t} H_{\text{int}} e^{-iH_0 t}. \quad (4.27)$$

The time-dependent operator $H_{\text{int}}(t)$ is defined in the interaction picture, and depends on the fields $\phi_{\text{in}}, \pi_{\text{in}}$ in the “in” basis. Let us now solve the equation of motion for $U(t)$ with the boundary condition $\lim_{t \to -\infty} U(t) = 1$. Integrating eq. (4.27) gives

$$\int_{-\infty}^{t} \frac{d}{dt_1} U(t_1) dt_1 = -i \int_{-\infty}^{t} H_{\text{int}}(t_1) U(t_1) dt_1$$

$$U(t) - U(-\infty) = -i \int_{-\infty}^{t} H_{\text{int}}(t_1) U(t_1) dt_1$$

$$\Rightarrow U(t) = 1 - i \int_{-\infty}^{t} H_{\text{int}}(t_1) U(t_1) dt_1. \quad (4.28)$$

The rhs. still depends on $U$, but we can substitute our new expression for $U(t)$ into the integrand, which gives

$$U(t) = 1 - i \int_{-\infty}^{t} H_{\text{int}}(t_1) \left\{ 1 - i \int_{-\infty}^{t_1} H_{\text{int}}(t_2) U(t_2) dt_2 \right\} dt_1$$

$$= 1 - i \int_{-\infty}^{t} H_{\text{int}}(t_1) dt_1 - \int_{-\infty}^{t} dt_1 H_{\text{int}}(t_1) \int_{-\infty}^{t_1} dt_2 H_{\text{int}}(t_2) U(t_2), \quad (4.29)$$

where $t_2 < t_1 < t$. This procedure can be iterated further, so that the $n$th term in the sum is

$$(-i)^n \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n H_{\text{int}}(t_1) H_{\text{int}}(t_2) \cdots H_{\text{int}}(t_n). \quad (4.30)$$

This iterative solution could be written in much more compact form, were it not for the fact that the upper integration bounds were all different, and that the ordering $t_n < t_{n-1} < \ldots < t_1 < t$ had to be obeyed. Time ordering is an important issue, since one has to ensure that the interaction Hamiltonians act at the proper time, thereby ensuring the causality of the theory. By introducing the time-ordered product of operators, one can use a compact notation, such that the resulting expressions still obey causality. The time-ordered product of two fields $\phi(t_1)$ and $\phi(t_2)$ is defined as

$$T\{\phi(t_1) \phi(t_2)\} = \begin{cases} \phi(t_1)\phi(t_2) & t_1 > t_2 \\ \phi(t_2)\phi(t_1) & t_1 < t_2 \end{cases} \equiv \theta(t_1 - t_2) \phi(t_1)\phi(t_2) + \theta(t_2 - t_1) \phi(t_2)\phi(t_1), \quad (4.31)$$

where $\theta$ denotes the step function. The generalisation to products of $n$ operators is obvious. Using time ordering for the $n$th term of eq. (4.30) we obtain

$$\frac{(-i)^n}{n!} \prod_{i=1}^{n} \int_{-\infty}^{t} dt_i T\{H_{\text{int}}(t_1) H_{\text{int}}(t_2) \cdots H_{\text{int}}(t_n)\}, \quad (4.32)$$

30
and since this looks like the $n$th term in the series expansion of an exponential, we can finally rewrite the solution for $U(t)$ in compact form as

$$U(t) = T \exp \left\{ -i \int_{-\infty}^{t} H_{\text{int}}(t') \, dt' \right\},$$

(4.33)

where the “$T$” in front ensures the correct time ordering.

### 4.3 S-matrix and Green’s functions

The $S$-matrix, which relates the “in” and “out” fields before and after the scattering process, can be written as

$$S = 1 + iT,$$

(4.34)

where $T$ is commonly called the $T$-matrix. The fact that $S$ contains the unit operator means that also the case where none of the particles scatter is encoded in $S$. On the other hand, the non-trivial case is described by the $T$-matrix, and this is what we are interested in. Let us consider again the scattering process depicted in Fig. 2. The $S$-matrix element in this case is

$$S_{\text{fi}} = \left\langle k_1, k_2, \ldots, k_n; \text{out} \left| p_1, p_2; \text{in} \right. \right\rangle$$

$$= \left\langle k_1, k_2, \ldots, k_n; \text{out} \left| a_{\text{in}}^\dagger (p_1) p_2; \text{in} \right. \right\rangle,$$

(4.35)

where $a_{\text{in}}^\dagger$ is the creation operator pertaining to the “in” field $\phi_{\text{in}}$. Our task is now to express $a_{\text{in}}^\dagger$ in terms of $\phi_{\text{in}}$, and repeat this procedure for all other momenta labelling our Fock states. As we shall see, this will relate the $S$-matrix element $S_{\text{fi}}$ to the vacuum expectation value of fields, i.e. a Green function, which can be computed in various ways.

The following identities will prove useful

$$\hat{a}^\dagger (p) = \int d^3 x \left\{ (\partial_0 e^{-iq \cdot x}) \phi(x) - e^{-iq \cdot x} (\partial_0 \phi(x)) \right\}$$

$$\equiv -i \int d^3 x e^{-iq \cdot x} \overleftarrow{\partial_0} \phi(x),$$

(4.36)

$$\hat{a} (p) = -i \int d^3 x \left\{ (\partial_0 e^{iq \cdot x}) \phi(x) - e^{iq \cdot x} (\partial_0 \phi(x)) \right\}$$

$$\equiv i \int d^3 x e^{iq \cdot x} \overrightarrow{\partial_0} \phi(x).$$

(4.37)

The $S$-matrix element can then be rewritten as

$$S_{\text{fi}} = -i \int d^3 x_1 e^{-ip_1 \cdot x_1} \overleftarrow{\partial_0} \left\langle k_1, \ldots, k_n; \text{out} \left| \phi_{\text{in}}(x_1) \right| p_2; \text{in} \right\rangle$$

$$= -i \lim_{t_1 \to -\infty} \int d^3 x_1 e^{-ip_1 \cdot x_1} \overleftarrow{\partial_0} \left\langle k_1, \ldots, k_n; \text{out} \left| \phi(x_1) \right| p_2; \text{in} \right\rangle,$$

(4.38)

where in the last line we have used eq. (4.4) to replace $\phi_{\text{in}}$ by $\phi$. We can now rewrite $\lim_{t_1 \to -\infty}$ using the following identity, which holds for an arbitrary, differentiable function $f(t)$, whose limit $t \to \pm \infty$ exists:

$$\lim_{t \to -\infty} f(t) = \lim_{t \to +\infty} f(t) - \int_{-\infty}^{+\infty} \frac{df}{dt} \, dt.$$
The \( S \)-matrix element then reads

\[
S_{\text{fi}} = -i \lim_{t_1 \to +\infty} \int d^3 x_1 e^{-ip_1 \cdot x_1} \frac{\partial}{\partial t_1} \left\langle \mathbf{k}_1, \ldots, \mathbf{k}_n; \text{out} \left| \phi(x_1) \right| p_2; \text{in} \right\rangle \\
+ i \int_{-\infty}^{+\infty} dt_1 \frac{\partial}{\partial t_1} \left\{ \int d^3 x_1 e^{-ip_1 \cdot x_1} \frac{\partial}{\partial t_0} \left\langle \mathbf{k}_1, \ldots, \mathbf{k}_n; \text{out} \left| \phi(x_1) \right| p_2; \text{in} \right\rangle \right\} .
\]

(4.40)

The first term in this expression involves \( \lim_{t_1 \to +\infty} \phi = \phi_{\text{out}} \), which gives rise to a contribution

\[
\propto \left\langle \mathbf{k}_1, \ldots, \mathbf{k}_n; \text{out} \left| a_{\text{out}}^\dagger (p_1) \right| p_2; \text{in} \right\rangle .
\]

(4.41)

This is non-zero only if \( p_1 \) is equal to one of \( \mathbf{k}_1, \ldots, \mathbf{k}_n \). This, however, means that the particle with momentum \( p_1 \) does not scatter, and hence the first term does not contribute to the \( T \)-matrix of eq. (4.34). We are then left with the following expression for \( S_{\text{fi}} \):

\[
S_{\text{fi}} = -i \int d^4 x_1 \left\langle \mathbf{k}_1, \ldots, \mathbf{k}_n; \text{out} \left| \partial_0 \left\{ \left( \partial_0 e^{-ip_1 \cdot x_1} \right) \phi(x_1) - e^{-ip_1 \cdot x_1} \left( \partial_0 \phi(x_1) \right) \right\} \right| p_2; \text{in} \right\rangle .
\]

(4.42)

The time derivatives in the integrand can be worked out:

\[
\partial_0 \left\{ \left( \partial_0 e^{-ip_1 \cdot x_1} \right) \phi(x_1) - e^{-ip_1 \cdot x_1} \left( \partial_0 \phi(x_1) \right) \right\}
= - \left[ E(p_1) \right] \left( e^{-ip_1 \cdot x_1} \phi(x_1) - e^{-ip_1 \cdot x_1} \partial_0 \phi(x_1) \right)
= - \left\{ \left( \left( -\nabla^2 + m^2 \right) e^{-ip_1 \cdot x_1} \right) \phi(x_1) + e^{-ip_1 \cdot x_1} \partial_0^2 \phi(x_1) \right\},
\]

(4.43)

where we have used that \( -\nabla^2 e^{-ip_1 \cdot x_1} = p_1^2 e^{-ip_1 \cdot x_1} \). For the \( S \)-matrix element one obtains

\[
S_{\text{fi}} = i \int d^4 x_1 e^{-ip_1 \cdot x_1} \left\langle \mathbf{k}_1, \ldots, \mathbf{k}_n; \text{out} \left| \left( \partial_0^2 - \nabla^2 + m^2 \right) \phi(x_1) \right| p_2; \text{in} \right\rangle
= i \int d^4 x_1 e^{-ip_1 \cdot x_1} \left( \Box x_1 + m^2 \right) \left\langle \mathbf{k}_1, \ldots, \mathbf{k}_n; \text{out} \left| \phi(x_1) \right| p_2; \text{in} \right\rangle .
\]

(4.44)

What we have obtained after this rather lengthy step of algebra is an expression in which the field operator is sandwiched between Fock states, one of which has been reduced to a one-particle state. We can now successively eliminate all momentum variables from the Fock states, by repeating the procedure for the momentum \( p_2 \), as well as the \( n \) momenta of the “out” state. The final expression for \( S_{\text{fi}} \) is

\[
S_{\text{fi}} = (i)^{n+2} \int d^4 x_1 \int d^4 x_2 \int d^4 y_1 \cdots \int d^4 y_n \, e^{-ip_1 \cdot x_1 - ip_2 \cdot x_2 + ik_1 \cdot y_1 + \ldots + k_n \cdot y_n}
\times \left( \Box x_1 + m^2 \right) \left( \Box x_2 + m^2 \right) \left( \Box y_1 + m^2 \right) \cdots \left( \Box y_n + m^2 \right)
\times \left\langle 0; \text{out} \left| T \{ \phi(y_1) \cdots \phi(y_n) \phi(x_1) \phi(x_2) \} \right| 0; \text{in} \right\rangle .
\]

(4.45)

where the time-ordering inside the vacuum expectation value (VEV) ensures that causality is obeyed. The above expression is known as the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula. It relates the formal definition of the scattering amplitude to a VEV of time-ordered fields. The VEV in the LSZ formula for the scattering of two initial particles into \( n \) particles in the final state is called the \( (n+2) \)-point Green’s function:

\[
G_{n+2}(y_1, y_2, \ldots, y_n, x_1, x_2) = \left\langle 0; \text{out} \left| T \{ \phi(y_1) \cdots \phi(y_n) \phi(x_1) \phi(x_2) \} \right| 0; \text{in} \right\rangle .
\]

(4.46)
In the next section we shall discuss a method how to compute the Green’s function of scalar field theory. This will allow for an explicit calculation of \( G_{n+2} \). The LSZ formula then provides the link to the scattering amplitude \( S_{\text{h}} \), and \( |S_{\text{h}}|^2 \) finally yields the cross section, which can be measured in an experiment.

Before we can tackle the actual computation of Green’s function, we must take a further step. Let us consider the \( n \)-point Green’s function

\[
G_n(x_1, \ldots, x_n) = \langle 0 | T \{ \phi(x_1) \cdots \phi(x_n) \} | 0 \rangle. \tag{4.47}
\]

The fields \( \phi \) which appear in this expression are Heisenberg fields, whose time evolution is governed by the full Hamiltonian \( H_0 + H_{\text{int}} \). In particular, the \( \phi \)'s are not the \( \phi_{\text{in}} \)'s, which we know how to handle. In order to be useful for actual computations, we first have to isolate the dependence of the fields on the interaction Hamiltonian. Let us recall the relation between the Heisenberg fields \( \phi(t) \) and the “in”-fields\(^2\)

\[
\phi(t) = U^{-1}(t) \phi_{\text{in}}(t) U(t). \tag{4.48}
\]

We now assume that the fields are properly time-ordered, i.e. \( t_1 > t_2 > \ldots > t_n \), so that we can forget about writing \( T(\cdots) \) everywhere. After inserting eq. (4.48) into the definition of \( G_n \) one obtains

\[
G_n = \langle 0 | U^{-1}(t_1) \phi_{\text{in}}(t_1) U(t_1) U^{-1}(t_2) \phi_{\text{in}}(t_2) U(t_2) \cdots \\
\times U^{-1}(t_n) \phi_{\text{in}}(t_n) U(t_n) | 0 \rangle. \tag{4.49}
\]

Now we introduce another time label \( t \) such that \( t \gg t_1 \) and \( -t \ll t_1 \). For the \( n \)-point function we now obtain

\[
G_n = \langle 0 | U^{-1}(t) \left\{ U(t) U^{-1}(t_1) \phi_{\text{in}}(t_1) U(t_1) U^{-1}(t_2) \phi_{\text{in}}(t_2) U(t_2) \cdots \\
\times U^{-1}(t_n) \phi_{\text{in}}(t_n) U(t_n) U^{-1}(-t) \right\} U(-t) | 0 \rangle. \tag{4.50}
\]

The expression in curly braces is now time-ordered by construction. An important observation at this point is that it involves pairs of \( U \) and its inverse, for instance

\[
U(t) U^{-1}(t_1) \equiv U(t, t_1). \tag{4.51}
\]

One can easily convince oneself that \( U(t, t_1) \) provides the net time evolution from \( t_1 \) to \( t \). We can now write \( G_n \) as

\[
G_n = \langle 0 | U^{-1}(t) T \left\{ \phi_{\text{in}}(t_1) \cdots \phi_{\text{in}}(t_n) U(t, t_1) U(t_1, t_2) \cdots U(t_n, -t) \right\} U(-t) | 0 \rangle. \tag{4.52}
\]

Let us now take \( t \to \infty \). The relation between \( U(t) \) and the \( S \)-matrix eq. (4.25), as well as the boundary condition eq. (4.26) tell us that

\[
\lim_{t \to \infty} U(-t) = 1, \quad \lim_{t \to \infty} U(t, -t) = S, \tag{4.53}
\]

\(^2\)Here and in the following we suppress the spatial argument of the fields for the sake of brevity.
which can be inserted into the above expression. We still have to work out the meaning of \( \langle 0|U^{-1}(\infty) \rangle \) in the expression for \( G_n \). In a paper by Gell-Mann and Low it was argued that the time evolution operator must leave the vacuum invariant (up to a phase), which justifies the ansatz
\[
\langle 0|U^{-1}(\infty) \rangle = K \langle 0|, \tag{4.54}
\]
with \( K \) being the phase. Multiplying this relation with \( |0\rangle \) from the right gives
\[
\langle 0|U^{-1}(\infty)|0\rangle = K \langle 0|0\rangle = K. \tag{4.55}
\]
Furthermore, Gell-Mann and Low showed that
\[
\langle 0|U^{-1}(\infty)|0\rangle = \frac{1}{\langle 0|U(\infty)|0\rangle}, \tag{4.56}
\]
which implies
\[
K = \frac{1}{\langle 0|S|0\rangle}. \tag{4.57}
\]
After inserting all these relations into the expression for \( G_n \) we obtain
\[
G_n(x_1, \ldots, x_n) = \frac{\langle 0| T \{ \phi_{\text{in}}(x_1) \cdots \phi_{\text{in}}(x_n) \} S |0\rangle}{\langle 0|S|0\rangle}. \tag{4.58}
\]
The \( S \)-matrix is given by
\[
S = T \exp \left\{ -i \int_{-\infty}^{+\infty} H_{\text{int}}(t) \, dt \right\}, \quad H_{\text{int}} = H_{\text{int}}(\phi_{\text{in}}, \pi_{\text{in}}), \tag{4.59}
\]
and thus we have finally succeeded in expressing the \( n \)-point Green's function exclusively in terms of the “in”-fields. This completes the derivation of a relation between the general definition of the scattering amplitude \( S_{\text{fi}} \) and the VEV of time-ordered “in”-fields. The link between the scattering amplitude and the underlying field theory is provided by the \( n \)-point Green’s function.

Problems

4.1 Using the definition \( U(t) = e^{iH_0 t} e^{-iHt} \) derive the equation of motion for \( U(t) \), eq. (4.27).

4.2 Derive eqs. (4.36) and (4.37).

5 Perturbation Theory

In this section we are going to calculate the Green’s functions of scalar quantum field theory explicitly. We will specify the interaction Lagrangian in detail and use an approximation known as perturbation theory. At the end we will derive a set of rules, which
represent a systematic prescription for the calculation of Green's functions, and can be easily generalised to apply to other, more complicated field theories. These are the famous Feynman rules.

We start by making a definite choice for the interaction Lagrangian $L_{\text{int}}$. Although one may think of many different expressions for $L_{\text{int}}$, one has to obey some basic principles: firstly, $L_{\text{int}}$ must be chosen such that the potential it generates is bounded from below – otherwise the system has no ground state. Secondly, our interacting theory should be renormalisable. Despite being of great importance, the second issue will not be addressed in these lectures. The requirement of renormalisability arises because if one computes quantities like the energy or charge of a particle, one typically obtains a divergent result\(^3\). There are classes of quantum field theories, called renormalisable, in which these divergences can be removed by suitable redefinitions of the fields and the parameters (masses and coupling constants).

For our theory of a real scalar field in four space-time dimensions, it turns out that the only interaction term which leads to a renormalisable theory must be quartic in the fields. Thus we choose

$$L_{\text{int}} = -\frac{\lambda}{4!} \phi^4(x), \quad (5.1)$$

where the coupling constant $\lambda$ describes the strength of the interaction between the scalar fields, much like, say, the electric charge describing the strength of the interaction between photons and electrons. The full Lagrangian of the theory then reads

$$\mathcal{L} = \mathcal{L}_0 + L_{\text{int}} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4, \quad (5.2)$$

and the explicit expressions for the interaction Hamiltonian and the $S$-matrix are

$$H_{\text{int}} = -L_{\text{int}}, \quad H_{\text{int}} = \frac{\lambda}{4!} \int d^4x \phi^4_{\text{in}}(x,t)$$

$$S = T \exp \left\{ -i \frac{\lambda}{4!} \int d^4x \phi^4_{\text{in}}(x) \right\}. \quad (5.3)$$

The $n$-point Green’s function is

$$G_n(x_1, \ldots, x_n) = \sum_{r=0}^\infty \left( -\frac{i \lambda}{4!} \right)^r \frac{1}{r!} \left\langle 0 \left| T \left\{ \phi_{\text{in}}(x_1) \cdots \phi_{\text{in}}(x_n) \left( \int d^4y \phi^4_{\text{in}}(y) \right)^r \right\} \right| 0 \right\rangle.$$  \quad (5.4)

In order to evaluate this expression we must expand $G_n$ in powers of the coupling $\lambda$. This only makes sense if $\lambda$ is sufficiently small. In other words, the interaction Lagrangian must act as a small perturbation on the system. As a consequence, the procedure of expanding Green’s functions in powers of the coupling is referred to as perturbation theory.

\(^3\)This is despite the subtraction of the vacuum energy discussed in section 3.
5.1 Wick’s Theorem

The \( n \)-point Green’s function in eq. (5.4) involves the time-ordered product over at least \( n \) fields. There is a method to express VEV’s of \( n \) fields, i.e. \( \langle 0|T\{\phi_{i_1}(x_1)\cdots\phi_{i_n}(x_n)\}|0 \rangle \) in terms of VEV’s involving two fields only. This is known as Wick’s theorem.

Let us for the moment ignore the subscript “in” and return to the definition of normal-ordered fields. The normal-ordered product \( :\phi(x_1)\phi(x_2): \) differs from \( \phi(x_1)\phi(x_2) \) by the vacuum expectation value, i.e.

\[
\phi(x_1)\phi(x_2) = :\phi(x_1)\phi(x_2) + \langle 0|\phi(x_1)\phi(x_2)|0 \rangle.
\]

We are now going to combine normal-ordered products with time ordering. The time-ordered product \( T\{\phi(x_1)\phi(x_2)\} \) is given by

\[
T\{\phi(x_1)\phi(x_2)\} = \phi(x_1)\phi(x_2)\theta(t_1 - t_2) + \phi(x_2)\phi(x_1)\theta(t_2 - t_1)
\]

\[
= :\phi(x_1)\phi(x_2) : \left( \theta(t_1 - t_2) + \theta(t_2 - t_1) \right)
\]

\[
+ \langle 0|\phi(x_1)\phi(x_2)\theta(t_1 - t_2) + \phi(x_2)\phi(x_1)\theta(t_2 - t_1)|0 \rangle.
\]

(5.6)

Here we have used the important observation that

\[
:\phi(x_1)\phi(x_2) : = :\phi(x_2)\phi(x_1) :,
\]

which means that normal-ordered products of fields are automatically time-ordered.\(^4\)

Equation (5.6) is Wick’s theorem for the case of two fields:

\[
T\{\phi(x_1)\phi(x_2)\} = :\phi(x_1)\phi(x_2) : + \langle 0|T\{\phi(x_1)\phi(x_2)\}|0 \rangle.
\]

(5.8)

For the case of three fields, Wick’s theorem yields

\[
T\{\phi(x_1)\phi(x_2)\phi(x_3)\} = :\phi(x_1)\phi(x_2)\phi(x_3) : + :\phi(x_1) : \langle 0|T\{\phi(x_2)\phi(x_3)\}|0 \rangle + \langle 0|T\{\phi(x_1)\phi(x_2)\}|0 \rangle
\]

\[
+ :\phi(x_2) : \langle 0|T\{\phi(x_1)\phi(x_3)\}|0 \rangle + \langle 0|T\{\phi(x_1)\phi(x_2)\}|0 \rangle
\]

(5.9)

At this point the general pattern becomes clear: any time-ordered product of fields is equal to its normal-ordered version plus terms in which pairs of fields are removed from the normal-ordered product and sandwiched between the vacuum to form 2-point functions. Then one sums over all permutations. Without proof we give the expression for the general case of \( n \) fields (\( n \) even):

\[
T\{\phi(x_1)\cdots\phi(x_n)\} =
\]

\[
:\phi(x_1)\cdots\phi(x_n) :
\]

\[
+ :\phi(x_1)\cdots\phi(x_i) \cdots \underbrace{\phi(x_j)\cdots\phi(x_n)}_{\text{perm.}} : \langle 0|T\{\phi(x_i)\phi(x_j)\}|0 \rangle + \text{perms.}
\]

\[
+ :\phi(x_1)\cdots\phi(x_i) \cdots \underbrace{\phi(x_j)\cdots\phi(x_k)}_{\text{perm.}}\cdots \phi(x_l) \cdots \phi(x_n) : 
\]

\[
\times \langle 0|T\{\phi(x_i)\phi(x_j)\}|0 \rangle \langle 0|T\{\phi(x_k)\phi(x_l)\}|0 \rangle + \text{perms.}
\]

\[
+ \ldots +
\]

\[
+ \langle 0|T\{\phi(x_1)\phi(x_2)\}|0 \rangle \langle 0|T\{\phi(x_3)\phi(x_4)\}|0 \rangle \cdots \langle 0|T\{\phi(x_{n-1})\phi(x_n)\}|0 \rangle
\]

\[
+ \text{perms}.\quad (5.10)
\]

\(^4\)The reverse is, however, not true!
The symbol $\overline{\phi(x_i)}$ indicates that $\phi(x_i)$ has been removed from the normal-ordered product.

Let us now go back to $\langle 0\vert T\{\phi(x_1)\cdots\phi(x_n)\}\vert 0\rangle$. If we insert Wick’s theorem, then we find that only the contribution in the last line of eq. (5.10) survives: by definition the VEV of a normal-ordered product of fields vanishes, and it is precisely the last line of Wick’s theorem in which no normal-ordered products are left. The only surviving contribution is that in which all fields have been paired or “contracted”. Sometimes a contraction is represented by the notation:

$$\phi(x_i)\phi(x_j) \equiv \langle 0\vert T\{\phi(x_i)\phi(x_j)\}\vert 0\rangle,$$

i.e. the pair of fields which is contracted is joined by the braces. Wick’s theorem can now be rephrased as

$$\langle 0\vert T\{\phi(x_1)\cdots\phi(x_n)\}\vert 0\rangle = \text{sum of all possible contractions of } n \text{ fields.} \tag{5.12}$$

Let us look at a few examples. The first is the 4-point function

$$\langle 0\vert T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}\vert 0\rangle = \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) + \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) + \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \tag{5.13}$$

The second example is again a 4-point function, where two of the fields are also normal-ordered:

$$\langle 0\vert T\{\phi(x_1)\phi(x_2) : \phi(x_3)\phi(x_4) :\}\vert 0\rangle = \phi(x_1)\phi(x_2) : \phi(x_3)\phi(x_4) : + \phi(x_1)\phi(x_2) : \phi(x_3)\phi(x_4) : \tag{5.14}$$

In this example, though, the contraction of : $\phi(x_3)\phi(x_4)$ : vanishes by construction, so only the last two terms survive! As a general rule, contractions which only involve fields inside a normal-ordered product vanish. Such contractions contribute only to the vacuum. Normal ordering can therefore simplify the calculation of Green’s functions quite considerably, as we shall see explicitly below.

### 5.2 The Feynman propagator

Using Wick’s Theorem one can relate any $n$-point Green’s functions to an expression involving only 2-point functions. Let us have a closer look at

$$G_2(x, y) = \langle 0\vert T\{\phi_{in}(x)\phi_{in}(y)\}\vert 0\rangle. \tag{5.15}$$

We can now insert the solution for $\phi$ in terms of $\hat{a}$ and $\hat{a}^\dagger$. If we assume $t_x > t_y$ then $G_2(x, y)$ can be written as

$$G_2(x, y) = \int \frac{d^3p \; d^3q}{(2\pi)^6 \; 4E(p)E(q)} \times \langle 0\vert (\hat{a}^\dagger(p) e^{ip\cdot x} + \hat{a}(p) e^{-ip\cdot x}) (\hat{a}^\dagger(q) e^{iq\cdot y} + \hat{a}(q) e^{-iq\cdot y}) \vert 0\rangle \tag{5.16}$$

$$= \int \frac{d^3p \; d^3q}{(2\pi)^6 \; 4E(p)E(q)} e^{-ip\cdot x + iq\cdot y} \langle 0\vert \hat{a}(p)\hat{a}^\dagger(q) \vert 0\rangle .$$
Figure 3: Scattering of two initial particles with momenta $p_1$ and $p_2$ into 2 particles with momenta $k_1$ and $k_2$.

This shows that $G_2$ can be interpreted as the amplitude for a meson which is created at $y$ and destroyed again at point $x$. We can now replace $\hat{a}(p)\hat{a}^\dagger(q)$ by its commutator:

$$G_2(x, y) = \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{2E(p)E(q)} e^{-i(p\cdot x + q\cdot y)} \langle 0 | [\hat{a}(p), \hat{a}^\dagger(q)] | 0 \rangle$$

and the general result, after restoring time-ordering, reads

$$G_2(x, y) = \int \frac{d^3p}{(2\pi)^3} 2E(p) e^{-i(p\cdot (x-y))}, \quad (5.17)$$

Furthermore, using contour integration one can show that this expression can be rewritten as a 4-dimensional integral

$$G_2(x, y) = i \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot (x-y)}}{p^2 - m^2 + i\epsilon}, \quad (5.19)$$

where $\epsilon$ is a small parameter which ensures that $G_2$ does not develop a pole. This calculation has established that $G_2(x, y)$ actually depends only on the difference $(x-y)$.

Equation (5.19) is called the Feynman propagator $G_F(x-y)$:

$$G_F(x-y) \equiv \langle 0 | T\{\phi(x)\phi(y)\} | 0 \rangle = i \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot (x-y)}}{p^2 - m^2 + i\epsilon}. \quad (5.20)$$

The Feynman propagator is a Green’s function of the Klein-Gordon equation, i.e. it satisfies

$$\left(\Box + m^2\right) G_F(x-y) = -i\delta^4(x-y), \quad (5.21)$$

and describes the propagation of a meson between the space-time points $x$ and $y$.

5.3 Two-particle scattering to $O(\lambda)$

Let us now consider a scattering process in which two incoming particles with momenta $p_1$ and $p_2$ scatter into two outgoing ones with momenta $k_1$ and $k_2$, as shown in Fig. 3.

The $S$-matrix element in this case is

$$S_{\text{fi}} = \langle k_1, k_2; \text{out}| p_1, p_2; \text{in} \rangle = \langle k_1, k_2; \text{in} | S| p_1, p_2; \text{in} \rangle, \quad (5.22)$$
and \( S = 1 + iT \). The LSZ formula eq. (4.45) tells us that we must compute \( G_4 \) in order to obtain \( S \). Let us work out \( G_4 \) in powers of \( \lambda \) using Wick’s theorem. To make life simpler, we shall introduce normal ordering into the definition of \( S \), i.e.

\[
S = T \exp \left\{ -i \frac{\lambda}{4!} \int d^4 x : \phi^4_{\text{in}}(x) : \right\}
\]  

(5.23)

Suppressing the subscripts “in” from now on, the expression we have to evaluate order by order in \( \lambda \) is

\[
G_n(x_1, \ldots, x_n)
\]

(5.24)

\[
= \sum_{r=0}^{\infty} \left( -i \frac{\lambda}{4!} \right)^r \frac{1}{r!} \left\langle 0 \left| T \left\{ \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \left( \int d^4 y : \phi^4(y) : \right)^r \right\} \right| 0 \right\rangle
\]

Starting with the denominator, we note that for \( r = 0 \) one finds

\[
r = 0 : \quad \text{denominator} = 1.
\]

(5.25)

If \( r = 1 \), then the expression in the denominator only involves fields which are normal-ordered. Following the discussion at the end of section 5.1 we conclude that these contributions must vanish, hence

\[
r = 1 : \quad \text{denominator} = 0.
\]

(5.26)

The contribution for \( r = 2 \), however, is non-zero. But then the case of \( r = 2 \) corresponds already to \( O(\lambda^2) \), which is higher than the order which we are working to. Therefore

\[
\text{denominator} = 1 \text{ to order } \lambda.
\]

(5.27)

Turning now to the numerator, we start with \( r = 0 \) and apply Wick’s theorem, which gives

\[
r = 0 : \quad \langle 0 | T \{ \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \} | 0 \rangle
\]

\[
= G_F(x_1 - x_2) G_F(x_3 - x_4) + G_F(x_1 - x_3) G_F(x_2 - x_4)
\]

\[
+ G_F(x_1 - x_4) G_F(x_2 - x_3),
\]

(5.28)

which can be graphically represented as

But this is the same answer as if we had set \( \lambda = 0 \), so \( r = 0 \) in the numerator does not describe scattering and is hence not a contribution to the \( T \)-matrix.
For \( r = 1 \) in the numerator we have to evaluate
\[
\begin{align*}
   r = 1 : & \quad - \frac{i\lambda}{4!} \left< 0 \left| T \left\{ \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) : \int d^4y \phi^4(y) : \right\} \right| 0 \right>
   & = - \frac{i\lambda}{4!} \int d^4y \, 4! \, G_F(x_1 - y)G_F(x_2 - y)G_F(x_3 - y)G_F(x_4 - y), \quad (5.29)
\end{align*}
\]
where we have taken into account that contractions involving two fields inside : \cdots : vanish. The factor \( 4! \) inside the integrand is a combinatorial factor: it is equal to the number of permutations which must be summed over according to Wick’s theorem and cancels the \( 4! \) in the denominator of the interaction Lagrangian. Graphically this contribution is represented by
\[
\begin{align*}
   -i\lambda & \int d^4y \\
   & \quad \begin{array}{c}
           \bullet \quad \bullet \\
           x_1 & x_3 \\
           \bullet \quad \bullet \\
           x_2 & x_4 \\
           \bullet \quad \bullet \\
           y
          \end{array}
\end{align*}
\]
where the integration over \( y \) denotes the sum over all possible locations of the interaction point \( y \). Without normal ordering we would have encountered the following contributions for \( r = 1 \):
\[
\begin{align*}
   & \quad + \\
   & \quad + \cdots
\end{align*}
\]
Such contributions are corrections to the vacuum and are cancelled by the denominator. This demonstrates how normal ordering simplifies the calculation by automatically subtracting terms which do not contribute to the actual scattering process.

To summarise, the final answer for the scattering amplitude to \( O(\lambda) \) is given by eq. (5.29).

5.4 Graphical representation of the Wick expansion: Feynman rules

We have already encountered the graphical representation of the expansion of Green’s functions in perturbation theory after applying Wick’s theorem. It is possible to formulate a simple set of rules which allow to draw the graphs directly without using Wick’s theorem and to write down the corresponding algebraic expressions.

We again consider a neutral scalar field whose Lagrangian is
\[
\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4. \quad (5.30)
\]
Suppose now that we want to compute the \( O(\lambda^m) \) contribution to the \( n \)-point Green’s function \( G_n(x_1, \ldots, x_n) \). This is achieved by going through the following steps:

1. Draw all distinct diagrams with \( n \) external lines and \( m \) 4-fold vertices:
• Draw $n$ dots and label them $x_1, \ldots, x_n$ (external points)
• Draw $m$ dots and label them $y_1, \ldots, y_m$ (vertices)
• Join the dots according to the following rules:
  – only one line emanates from each $x_i$
  – exactly four lines run into each $y_j$
  – the resulting diagram must be connected, i.e. there must be a continuous path between any two points.

2) Assign a factor $\frac{-i\lambda}{4!} \int d^4 y_i$ to the vertex at $y_i$.

3) Assign a factor $G_F(x_i - y_j)$ to the line joining $x_i$ and $y_j$.

4) Multiply by the number of contractions $C$ from the Wick expansion which lead to the same diagram.

These are the Feynman rules for scalar field theory in position space.

Let us look at an example, namely the 2-point function. According to the Feynman rules the contributions up to order $\lambda^2$ are as follows:

$O(1): \quad x_1 \rightarrow x_2 = G_F(x_1 - x_2)$

$O(\lambda): \quad x_1 \rightarrow y \rightarrow x_2 \quad \Rightarrow \quad \text{tadpole diagram};$
\hspace{1cm} (cancelled by normal ordering)

$O(\lambda^2): \quad x_1 \rightarrow y_1 \rightarrow y_2 \rightarrow x_2 \quad \Rightarrow \quad \text{tadpole diagram};$
\hspace{1cm} (cancelled by normal ordering)

$O(\lambda^2): \quad x_1 \rightarrow y_1 \rightarrow y_2 \rightarrow x_2$
\hspace{1cm} $= C \left( \frac{-i\lambda}{4!} \right)^2 \int d^4 y_1 d^4 y_2 \ G_F(x_1 - y_1) [G_F(y_1 - y_2)]^3 G_F(y_2 - x_2)$

The combinatorial factor for this contribution is worked out as $C = 4 \cdot 4!$. Note that the same graph, but with the positions of $y_1$ and $y_2$ interchanged is topologically distinct. Numerically it has the same value as the above graph, and so the corresponding expression has to be multiplied by a factor 2.

Another contribution at order $\lambda^2$ is
This contribution must be discarded, since not all of the points are connected via a continuous line.

Let us end this discussion with a small remark on the tadpole diagrams encountered above. These contributions to the 2-point function are cancelled if the interaction term is normal-ordered. However, unlike the case of the 4-point function, the corresponding diagrams satisfy the Feynman rules listed above. In particular, the diagrams are connected and are not simply vacuum contributions. They must hence be included in the expression for the 2-point function.

5.5 Feynman rules in momentum space

It is often simpler to work in momentum space, and hence we will discuss the derivation of Feynman rules in this case. If one works in momentum space, the Green’s functions are related to those in position space by a Fourier transform

\[
G_n(x_1, \ldots, x_n) = \int \frac{d^4p_1}{(2\pi)^4} \cdots \int \frac{d^4p_n}{(2\pi)^4} e^{ip_1 \cdot x_1 + \cdots + ip_n \cdot x_n} G_n(p_1, \ldots, p_n).
\]

The Feynman rules then serve to compute the Green’s function \( \tilde{G}_n(p_1, \ldots, p_n) \) order by order in the coupling.

In every scattering process the overall momentum must be conserved, and hence

\[
\sum_{i=1}^{n} p_i = 0.
\]

This can be incorporated into the definition of the momentum space Green’s function one is trying to compute:

\[
\tilde{G}_n(p_1, \ldots, p_n) = (2\pi)^4 \delta^4 \left( \sum_{i=1}^{n} p_i \right) G_n(p_1, \ldots, p_n).
\]

Here we won’t be concerned with the exact derivation of the momentum space Feynman rules, but only list them as a recipe.

**Feynman rules (momentum space)**

1. Draw all distinct diagrams with \( n \) external lines and \( m \) 4-fold vertices:
   - Assign momenta \( p_1, \ldots, p_n \) to the external lines
   - Assign momenta \( k_j \) to the internal lines
(2) Assign to each external line a factor

\[ \frac{i}{p_i^2 - m^2 + i\epsilon} \]

(3) Assign to each internal line a factor

\[ \int \frac{d^4k_j}{(2\pi)^4} \frac{i}{k_j^2 - m^2 + i\epsilon} \]

(4) Each vertex contributes a factor

\[ -\frac{i\lambda}{4!} (2\pi)^4 \delta^4 \left( \sum \text{momenta} \right) , \]

(the delta function ensures that momentum is conserved at each vertex).

(5) Multiply by the combinatorial factor \( C \), which is the number of contractions leading to the same momentum space diagram (note that \( C \) may be different from the combinatorial factor for the same diagram considered in position space!)

### 5.6 \( S \)-matrix and truncated Green’s functions

The final topic in these lectures is the derivation of a simple relation between the \( S \)-matrix element and a particular momentum space Green’s function, which has its external legs amputated: the so-called truncated Green’s function. This further simplifies the calculation of scattering amplitudes using Feynman rules.

Let us return to the LSZ formalism and consider the scattering of \( m \) initial particles (momenta \( p_1, \ldots, p_m \)) into \( n \) final particles with momenta \( k_1, \ldots, k_n \). The LSZ formula tells us that the \( S \)-matrix element is given by

\[
\langle k_1, \ldots, k_n; \text{out} | p_1, \ldots, p_n; \text{in} \rangle = (i)^{n+m} \int \prod_{i=1}^{m} d^4x_i \int \prod_{j=1}^{n} d^4y_j \exp \left\{ -i \sum_{i=1}^{m} p_i \cdot x_i + i \sum_{j=1}^{n} k_j \cdot y_j \right\}
\times \prod_{i=1}^{m} (\Box_{x_i} + m^2) \prod_{j=1}^{n} (\Box_{y_j} + m^2) G_{n+m}(x_1, \ldots, x_m, y_1, \ldots, y_n). \tag{5.34}
\]

Let us have a closer look at \( G_{n+m}(x_1, \ldots, x_m, y_1, \ldots, y_n) \). As shown in Fig. 4 it can be split into Feynman propagators, which connect the external points to the vertices at \( z_1, \ldots, z_{n+m} \), and a remaining Green’s function \( \overline{G}_{n+m} \), according to

\[
G_{n+m} = \int d^4z_1 \cdots d^4z_{n+m} G_F(x_1 - z_1) \cdots G_F(y_n - z_{n+m}) \overline{G}_{n+m}(z_1, \ldots, z_{n+m}), \tag{5.35}
\]

where, perhaps for obvious reasons, \( \overline{G}_{n+m} \) is called the truncated Green’s function.

Putting eq. (5.35) back into the LSZ expression for the \( S \)-matrix element, and using that

\[
(\Box_{x_i} + m^2) G_F(x_i - z_i) = -i\delta^4(x_i - z_i) \tag{5.36}
\]

43
one obtains

\[ \langle k_1, \ldots, k_n; \text{out}|p_1, \ldots, p_m; \text{in} \rangle \]

\[ = (i)^{n+m} \int \prod_{i=1}^{m} d^4x_i \int \prod_{j=1}^{n} d^4y_j \exp \left\{ -i \sum_{i=1}^{m} p_i \cdot x_i + i \sum_{j=1}^{n} k_j \cdot y_j \right\} \]

\[ \times (-i)^{n+m} \int d^4z_1 \cdots d^4z_{n+m} \delta^4(x_1 - z_1) \cdots \delta^4(y_n - z_{n+m}) \overline{G}_{n+m}(z_1, \ldots, z_{n+m}). \]

After performing all the integrations over the \( z_k \)'s, the final relation becomes

\[ \langle k_1, \ldots, k_n; \text{out}|p_1, \ldots, p_m; \text{in} \rangle \]

\[ = \int \prod_{i=1}^{m} d^4x_i \prod_{j=1}^{n} d^4y_j \exp \left\{ -i \sum_{i=1}^{m} p_i \cdot x_i + i \sum_{j=1}^{n} k_j \cdot y_j \right\} \]

\[ \times \overline{G}_{n+m}(x_1, \ldots, x_m, y_1, \ldots, y_n) \]

\[ = \overline{G}_{n+m}(p_1, \ldots, p_m, k_1, \ldots, k_n), \]

where \( \overline{G}_{n+m} \) is the truncated \( n + m \)-point function in momentum space. This result shows that the scattering matrix element is directly given by the truncated Green’s function in momentum space. The latter can be obtained using the Feynman rules without the expression for the external legs.

Problems

5.1 Verify that

\[ : \phi(x_1)\phi(x_2) : = : \phi(x_2)\phi(x_1) : \]

**Hint:** write \( \phi = \phi^+ + \phi^- \), where \( \phi^+ \) and \( \phi^- \) are creation and annihilation components of \( \phi \).

5.2 Verify that

\[ G_F(x-y) = i \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip \cdot (x-y)}}{p^2 - m^2 + i\epsilon} \]

is a Green’s function of \( (\partial^\mu \partial_\mu + m^2) \) as \( \epsilon \to 0 \) (where \( \partial_\mu \equiv \partial / \partial x^\mu \)).

5.3 Find the expressions corresponding to the following momentum space Feynman diagrams
Integrate out all the $\delta$-functions but do not perform the remaining integrals.

6 Concluding remarks

Although we have missed out on many important topics in Quantum Field Theory, we got to the point where we established contact between the underlying formalism of Quantum Field theory and the Feynman rules, which are widely used. The main concepts of the formulation were discussed: we introduced field operators, multi-particle states that live in Fock spaces, creation and annihilation operators, as well as tools like normal ordering, all of which served to overcome the problems we encountered when we naively tried to write down a relativistic quantum theory. The basic ingredients to formulate a quantum theory for a real scalar field can be used to quantise the electron and photon fields. This is discussed in the lectures by Robert Thorne at this school. Renormalisation is a topic which is not so easily discussed in a relatively short period of time, and hence I refer the reader to standard textbooks on Quantum Field Theory, which are listed below. The same applies to the method of quantisation via path integrals.

Acknowledgements

I would like to thank Julia Sedgbeer for the invitation and for all her efforts which made the school a very pleasant experience. I would also like to thank Margaret Evans for her most friendly and efficient way of organising such an event. Many thanks go to my fellow lecturers, the tutors and all the students, whose persistent questions lead to many clarifications and improvements.

References


A Notation and conventions

4-vectors:
\[ x^\mu = (x^0, \vec{x}) = (t, \vec{x}) \]
\[ x_\mu = g_{\mu\nu} x^\nu = (x^0, -\vec{x}) = (t, -\vec{x}) \]

Metric tensor:
\[ g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]

Scalar product:
\[ x^\mu x_\mu = x^0 x_0 + x^1 x_1 + x^2 x_2 + x^3 x_3 = t^2 - x^2 \]

Gradient operators:
\[ \partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left( \frac{\partial}{\partial t}, -\nabla \right) \]
\[ \partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left( \frac{\partial}{\partial t}, \nabla \right) \]

\( \text{d’Alembertian: } \partial^\mu \partial_\mu = \frac{\partial^2}{\partial t^2} - \nabla^2 \equiv \Box \)

Momentum operator:
\[ \hat{p}^\mu = i\hbar \partial^\mu = \left( i\hbar \frac{\partial}{\partial t} - i\hbar \nabla \right) = \left( \hat{E}, \hat{\vec{p}} \right) \text{ (as it should be)} \]

\( \delta \)-functions:
\[ \int d^3p \, f(p) \, \delta^3(p - q) = f(q) \]
\[ \int d^3x \, e^{-i\vec{p} \cdot \vec{x}} = (2\pi)^3 \delta^3(\vec{p}) \]
\[ \int \frac{d^3p}{(2\pi)^3} \, e^{-i\vec{p} \cdot \vec{x}} = \delta^3(\vec{x}) \]

(similarly in four dimensions)

Note:
\[ \delta(x^2 - x_0^2) = \delta((x - x_0)(x + x_0)) = \frac{1}{2x} \{ \delta(x - x_0) + \delta(x + x_0) \} \]